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Informational steady-states and conditional entropy production in continuously monitored systems: the case of Gaussian systems

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We put forth a unifying formalism for the description of the thermodynamics of continuously monitored systems, where measurements are only performed on the environment connected to a system. We show, in particular, that the conditional and unconditional entropy production, which quantify the degree of irreversibility of the open system's dynamics, are related to each other by the Holevo quantity. This, in turn, can be further split into an information gain rate and loss rate, which provide conditions for the existence of *informational steady-states* (ISSs), i.e. stationary states of a conditional dynamics that are maintained owing to the unbroken acquisition of information. We illustrate the applicability of our framework through several examples, involving both discrete and continuous-variable systems. We also demonstrate the practical usefulness of our theory by providing a detailed account of a recent optomechanical experiment [Phys. Rev. Lett, **125**, 080601 (2020)].

I. INTRODUCTION

The dynamics of a quantum system depends not only on itself, but also on how it is probed, showcasing the remarkable extrinsic character of quantum mechanics. This unavoidable backaction due to measurements can be directly probed in the laboratory [1–4], and is by far the most intriguing and dramatic aspect of quantum theory. It also has a clear thermodynamic flavor [5], since backaction is an intrinsically irreversible process. A comprehensive theory describing the thermodynamics of monitored systems would therefore greatly benefit our understanding of the interplay between information and dissipation. Constructing such a theory, however, is not trivial, since it requires reformulating the 2nd law to take into account the information learned from the measurements. We call this a *conditional 2nd law*. It quantifies which processes are allowed, *given* a certain set of measurement outcomes. Interestingly, due to measurement backaction, the noise introduced by the measurement can actually make the conditional process more irreversible, as recently demonstrated in a superconducting qubit experiment [6].

When a system is coupled to two baths at different temperatures, it usually tends to a non-equilibrium steady-state (NESS), where the competition between the two baths keeps the system away from equilibrium. Continuous measurements can lead to a similar effect. In this case, noise is constantly being introduced by the environment or the measurement backaction. But information is also constantly being acquired. These two effects compete, leading the system toward an *informational steady-state* (ISS). Crucially, the ISS relies on the experimenter's knowledge of the measurement records. A beautiful experimental illustration of this effect was recently given in [7], where the authors studied an optomechanical membrane monitored by an optical field. By measuring the field, one could monitor the position of the mechanical membrane and thus infer a steady-state which was close to the ground state. Conversely, if the measurements are not read,

the membrane is perceived to be in a thermal state with higher temperatures. The ISS is therefore *colder*, due to the information acquired from the continuous measurement.

ISSs are just one example of the many interesting phenomena that emerge when quantum measurements are introduced in a thermodynamic picture. The deep connections between the two concepts, together with recent experimental advances in controlled quantum platforms, have led to a surge of interest in formulating conditional laws of thermodynamics [8–20]. This also motivated ground-breaking experiments applying these ideas to Maxwell demon engines and feedback control [6, 21–24]. In all these frameworks, however, the measurements are assumed to act directly in the system, making them explicitly invasive.

Conversely, our interest in this paper will be on formulating the laws of thermodynamics when the measurements are done only on the environment and *only* after it interacted with the system. The scenario is therefore non-invasive by construction, so that any information acquired can only make the process more reversible, even if the measurement is very poor (as is often the case when dealing with large environments). This represents a change in philosophy compared to, e.g., Ref. [12], where the measurement was introduced by coupling the system to a memory and then measuring the memory. In that case one constructs the conditional 2nd law by comparing the situation where the system is fully isolated, with that in which it is open due to the interaction with a memory. In our case, we assume instead that the interaction between system and bath is inevitable and will happen whether or not we measure it. We then ask how measuring the bath affects the degree of irreversibility of the process.

Crucially, the framework we develop will focus on continuously monitored system, in contrast to e.g. Ref [12]. It is therefore particularly suited for describing ISSs. Our endeavor began in Ref. [20], where we put forth a semiclassical theory valid for Gaussian processes. We were interested in quantum optical experiments, which have already been using some of these ideas for many decades, in the framework of continuously monitored systems [25, 26]. In fact, our theory was recently employed in [27] to experimentally assess the

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conditional 2nd law in an optomechanical system. However, in addition to being semiclassical, the framework of Ref. [20] also has another serious limitation: it is formulated solely in terms of the stochastic master equation obeyed by the system; that is, it does not require an explicit model of the environment, but only which type of open dynamics it produces.

There has been increasing evidence that a proper formulation of thermodynamics in the quantum regime is only possible if information on the environment and the system-environment interactions are provided [28]. Reduced descriptions, based only on master equations, can show apparent violations of the 2nd law [29], something which can only be resolved by introducing a specific model of the environment [30]. Our efforts in modeling the experiments in [27] also strongly corroborate this view. In fact, an explicit illustration will be provided below, where we will show that models that yield the same type of master equation can have completely different thermodynamics features.

In this paper we put forth a very general framework for describing the thermodynamics of continuously monitored systems, where measurements are only done indirectly in the bath. The formalism applies to a broad variety of systems and process, and is particularly suited for describing ISSs. The building block we use is to replace the continuous dynamics by a stroboscopic evolution in small time-steps, described in terms of a collisional model (CM) [31–40]. This has two main advantages. First, the thermodynamics of CMs is by now very well understood [30, 40–43] (see also [28] for a recent review). And second, CMs naturally emerge in quantum optics, from a discretization of the field operator into discrete time-bins [44, 45]. The typical scenario is a system interacting with an optical cavity, where a constant flow of photons is injected by an external pump [cf. Fig. 1(a)]. At each time step, the system will only interact with a certain time-window of the input/output field, thus transforming the dynamics into that of a series of sequential collisions between the system and some ancilla. Due to this connection, collisional models serve as a convenient tool for constructing the framework of continuous measurements in experimentally relevant systems. We shall refer to these as Continuously Monitored Collisional Models (CM²).

Our paper is organized as follows. Sec. II establishes the basic framework, including the collisional setup whose information flows and thermodynamic features are characterized in Sec. ???. We provide the assessment of some illustrative applications in Sec. ???, while Sec. III is dedicated to the discussion of Gaussian processes, using the detailed assessment of the experiment reported in Ref. [27] as a physically motivated case study. Finally, in Sec. V we draw our conclusions and highlight the perspectives opened by our approach.

II. INFORMATION AND THERMODYNAMICS OF CONTINUOUSLY MEASURED COLLISIONAL MODELS

In this Section we discuss the elements underpinning the CM² construction and the analysis of its informational and thermodynamic features. This will help us setting the con-

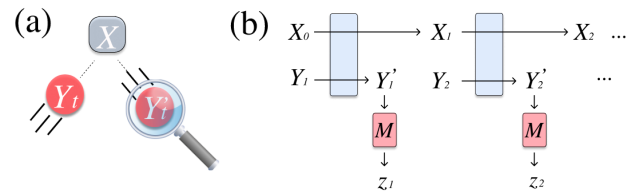


FIG. 1. (a) In a collisional model picture, the monitoring is introduced instead through a series of sequential collisions between the system X and independent ancillae Y_t , which are subjected to measurement after each collision. (b) Circuit representation of the model. The system is described stroboscopically (discrete time) by a state ρ_{X_t} . At each instant of time, it interacts with an independent ancilla, prepared in state Y_t , according to the map in Eq. (1). Afterwards, the ancillae are measured, as described by generalized measurement operators $\{M_z\}$, which produce a classical (and random) outcome z_t .

text of the study – dedicated to Gaussian bosonic systems – reported in Secs. III and IV. An extended discussion on the topic of this Section is in Ref. [46] to which we refer for further details.

A. Construction of CM²

we start by providing the key ingredients of the framework put forward in Ref. [46] to set up the CM² approach. We consider a system X with initial density matrix ρ_{X_0} interacting sequentially with independent and identically prepared (iid) ancillae Y_1, Y_2, \dots , all prepared always in the same state $\rho_{Y_t} = \rho_Y$. We discretize time and label the corresponding units as $t = 0, 1, 2, 3, \dots$. A unitary U_t acting on X and Y_t (a collision) takes the system from time $t - 1$ to t as

$$\rho_{X_t Y_t} = U_t(\rho_{X_{t-1}} \otimes \rho_{Y_t})U_t^\dagger, \quad (1)$$

where Y'_t refers to the state of ancilla Y_t after the event. Information on the state of the system is acquired by measuring the state $\rho'_{Y'_t}$ of the ancilla after the collision. Such measurement is described by a set of generalized measurement operators $\{M_z\}$, satisfying $\sum_z M_z^\dagger M_z = \mathbb{1}$ and associated probability for outcome z_t

$$P(z_t) = \text{tr} \left\{ M_{z_t} \rho'_{Y'_t} M_{z_t}^\dagger \right\}. \quad (2)$$

After its respective collision, ancilla Y'_t will not participate again in the dynamics, while a fresh ancilla Y_{t+1} is introduced and the sequence repeated. This allows us to build a set of time-ordered measurement records

$$\zeta_t = (z_1, \dots, z_t). \quad (3)$$

In a sense, ζ embodies the “integrated” information on X , while z_t is a *differential information gain* associated only with the step $X_{t-1} \rightarrow X_t$.

The joint distribution $P(\zeta_t)$ is given by [64]

$$P(\zeta_t) = \text{tr}_{XY_1 \dots Y_t} \left\{ M_{z_t} \dots M_{z_1} \rho_{XY_1 \dots Y_t} M_{z_1}^\dagger \dots M_{z_t}^\dagger \right\}, \quad (4)$$

where

$$\rho_{XY_1\dots Y_t} = \left(\prod_{k=1}^t U_k \right) \left(\rho_{X_0} \bigotimes_{j=1}^t \rho_{Y_j} \right) \left(\prod_{k=1}^t U_k \right)^\dagger.$$

It is now convenient to introduce the outcome-indexed completely positive trace non-preserving map

$$\mathcal{E}_z(\rho_X) = \text{tr}_Y \left\{ M_z U (\rho_X \otimes \rho_Y) U^\dagger M_z^\dagger \right\}, \quad (5)$$

which allows us to define the unnormalized conditional states

$$\varrho_{X_t|\zeta_t} = \mathcal{E}_{z_t}(\varrho_{X_{t-1}|\zeta_{t-1}}) \quad (6)$$

with initial condition $\varrho_{X_0|\zeta_0} = \rho_{X_0}$. One may readily verify that $\text{tr}_X \varrho_{X_t|\zeta_t} = \text{tr}_X \left\{ \mathcal{E}_{z_t} \circ \dots \circ \mathcal{E}_{z_1}(\rho_{X_0}) \right\} = P(\zeta_t)$. The states $\varrho_{X_t|\zeta_t}$ therefore contain the outcome distribution $P(\zeta_t)$ at any given time. Ref. [46] describes in detail how to interpret a CM² as a Hidden Markov model [9, 47, 48].

B. Informational aspects of CM²

The mismatch between the information carried by the conditional state of the system at time t and the corresponding unconditional one is quantified by the Holevo information [49]

$$I(X_t : \zeta_t) := S(X_t) - S(X_t|\zeta_t) = \sum_{\zeta_t} P(\zeta_t) D(\rho_{X_t|\zeta_t} \| \rho_{X_t}) \geq 0, \quad (7)$$

where $S(X_t) = -\text{tr} \rho_{X_t} \ln \rho_{X_t}$ is the von Neumann entropy of ρ_{X_t} , $S(X_t|\zeta_t) = \sum_{\zeta_t} P(\zeta_t) S(\rho_{X_t|\zeta_t})$ is the quantum-classical conditional entropy average over all trajectories set by the sequence of collisions, and $D(\rho \| \sigma) = \text{tr}(\rho \ln \rho - \rho \ln \sigma)$ is the quantum relative entropy. Eq. (7) thus provides a measure of the information known about the system given the measurements performed on the ancillae, which can be interpreted as the weighted average of the distance between $\rho_{X_t|\zeta_t}$ and ρ_{X_t} [46].

While Eq. (7) reflects the integrated information, acquired about the system, up to time t , the conditional Holevo information

$$G_t := I_c(X_t : z_t|\zeta_{t-1}) = I(X_t : \zeta_t) - I(X_t : \zeta_{t-1}) = S(X_t|\zeta_{t-1}) - S(X_t|\zeta_t) \quad (8)$$

quantifies the differential information gain obtained from a single outcome z , at each step.

We then define the differential information loss term

$$L_t := I(X_{t-1} : \zeta_{t-1}) - I(X_t : \zeta_{t-1}), \quad (9)$$

which can be shown to be strictly non-negative [46]. It is then immediate to prove that the trade-off between the gain in information and the (non-negative) measurement backaction quantified by the *information rate* $\Delta I_t := I(X_t : \zeta_t) - I(X_{t-1} : \zeta_{t-1})$ can be split as

$$\Delta I_t = G_t - L_t. \quad (10)$$

In the long-time limit, the system may reach a steady-state such that $\Delta I_\infty = 0$, although this might be associated with (mutually balancing) non-null information gain and loss rates. When this happens, we say the system has relaxed to an informational steady-state (ISS) such that

$$\Delta I_{\text{ISS}} = 0 \quad \text{but} \quad G_{\text{ISS}} = L_{\text{ISS}} \neq 0. \quad (11)$$

In an ISS, information is continuously acquired and balanced by the noise that is introduced by the measurement. Crucially, the ISS does *not* mean that $\rho_{X_t|\zeta_t}$ is no longer changing.

C. Thermodynamic aspects of CM²

The second law of thermodynamics ascribes the change of entropy in the state of the system occurring after each collision to the entropy flow between system and ancilla and a contribution representing the entropy that was irreversibly produced in the process. In formal terms

$$S(X_t) - S(X_{t-1}) = \Delta \Sigma_t^u - \Delta \Phi_t^u, \quad (12)$$

where $\Delta \Phi_t^u$ is the flow rate of entropy *from* the system *to* the ancilla in each collision, and $\Delta \Sigma_t^u$ is the rate of entropy produced in the process. In thermal processes, the entropy flow $\Delta \Phi_t^u$ is typically linked to the heat flow \dot{Q}_t entering the ancillae through Clausius' expression [52] $\Delta \Phi_t^u = \beta \dot{Q}_t$, where β is the inverse temperature of the thermal state the ancillae are in. By fixing $\Delta \Phi_t^u$ we then also fix $\Delta \Sigma_t^u$. This, however, only holds for thermal ancillae, thus restricting the range of applicability of the formalism.

Instead, we approach the problem using the framework of Ref. [53] (see also [40, 54]), which formulates the entropy production rate in information theoretic terms, as

$$\Delta \Sigma_t^u = I(X_t : Y'_t) + D(Y'_t \| Y_t) \geq 0, \quad (13)$$

where $I(X_t : Y'_t) = S(\rho_{X_t}) + S(\rho'_{Y'_t}) - S(\rho_{X_t Y'_t})$ is the quantum mutual information between system and ancilla after Eq. (1) and $D(Y'_t \| Y_t) = D(\rho_{Y'_t} \| \rho_{Y_t})$ is the relative entropy between the state of the ancilla before and after the collision. Irreversibility, from the perspective of the system, stems from tracing over the ancillae after the interaction in such a way that all quantities related either to the local state of the ancilla, or to their global correlations, are irretrievable [54].

In Ref. [46] it was proven that the entropy flux is depends solely on the degrees of freedom of the ancilla according to the general expression

$$\Delta \Phi_t^u = \sum_{j=1}^N \Delta \Phi_{tj}^u = \sum_{j=1}^N \text{tr} \left\{ (\rho_{Y_{tj}} - \rho_{Y_{tj}}) \ln \rho_{Y_{tj}} \right\}, \quad (14)$$

which states the additivity of the entropy flux, a property allowing one to compute the flux associated to each dissipation channel acting on the system.

Eqs. (12)-(14) specify the thermodynamics of the unconditional trajectories ρ_{X_t} , when no information about the ancillae

is recorded. We now aim to find a similar relation for the conditional trajectories $\rho_{X_t|\zeta_t}$. In this case, the relevant entropy is the quantum-classical conditional entropy $S(X_t|\zeta_t)$. We thus postulate a splitting akin to Eq. (12), i.e.

$$S(X_t|\zeta_t) - S(X_{t-1}|\zeta_{t-1}) = \Delta\Sigma_t^c - \Delta\Phi_t^c, \quad (15)$$

where $\Delta\Sigma_t^c$ and $\Delta\Phi_t^c$ are the conditional counterparts of the unconditional quantities used above. Following Ref. [46], and requesting that $\Delta\Phi_t^c$ depends only on quantities pertaining to the specific ancilla Y_t , one gets

$$\Delta\Phi_t^c = \text{tr} \left\{ (\rho_{Y_t} - \tilde{\rho}_{Y_t}) \ln \rho_{Y_t} \right\}, \quad (16)$$

where $\tilde{\rho}_{Y_t} = \sum_{z_t} P(z_t) \rho_{Y_t|z_t} = \sum_{z_t} M_{z_t} \rho_{Y_t} M_{z_t}^\dagger$ is the reconstructed state of the ancilla after the measurement. This expression shows that, depending on the measurement strategy $\{M_z\}$ being adopted, one in general has $\rho_{Y_t} \neq \tilde{\rho}_{Y_t}$, which in turns results in $\Delta\Phi_t^u \neq \Delta\Phi_t^c$, thus demonstrating the invasive nature of the measurements on the ancilla.

The comparison between Eq. (12) and Eq. (15), allows one to establish the relation between conditional and unconditional entropy production [46]

$$\Delta\Sigma_t^c = \Delta\Sigma_t^u - \Delta I_t. \quad (17)$$

Eq. (17) states that the act of conditioning the dynamics on the measurement outcome changes the entropy production by a quantity associated with the change in the Holevo information, thus connecting explicitly the information rates and thermodynamics.

The *differential* – i.e. referred to a single collision – conditional and unconditional entropy production $\Delta\Sigma_t^u$ and $\Delta\Sigma_t^c$ can be used to define corresponding integral quantities $\Sigma_t^\alpha = \sum_{\tau=1}^t \Delta\Sigma_\tau^\alpha$ ($\alpha = u, c$), which are in mutual relation as

$$\Sigma_t^c = \Sigma_t^u - I(X_t : \zeta_t). \quad (18)$$

This relation shows very clearly that the difference between conditional and unconditional irreversibility up to time t is strictly related to the net information $I(X_t : \zeta_t) \geq 0$. It thus follows that

$$\Sigma_t^u \geq \Sigma_t^c, \quad (19)$$

stating that – as the indirect measurement approach considered here does not result in direct backaction on the system – the act of conditioning reduces the irreversibility of a process. The net mismatch between Σ^u and Σ^c is actually bounded by the total information gain $\Sigma_u - \Sigma_c \geq \sum_{\tau=1}^t G_\tau$. The reduction in entropy production is thus at least the total gain. Conversely, one can view this as a bound on the minimal gain associated to a given entropy production mismatch.

III. CONTINUOUS VARIABLE SYSTEMS

Bosonic systems offer an essential platform for the implementation of continuous measurements, a scenario that is frequently found in quantum optical experiments. In this context,

the extensively developed toolbox of continuously monitored Gaussian processes [56–59] can be employed to build an insightful and simple formalism. Gaussian scenarios also allow for a more direct comparison with classical models, described in terms of Langevin or Fokker-Planck equations [60]. Indeed, similar considerations on the role of information in thermodynamics have been discussed in this classical context in Ref. [61].

A. Gaussian CM²s

We begin by reviewing the formalism developed in Ref. [59] for describing the unconditional and conditional dynamics. The system is described by N_X canonically conjugated operators $\hat{R}_X = (q_1, p_1, \dots, q_{N_X}, p_{N_X})$, while each ancilla is modeled by N_Y variables $\hat{R}_Y = (Q_1, P_1, \dots, Q_{N_Y}, P_{N_Y})$. Each collision is assumed to last for a small time dt and is ruled by a quadratic interaction Hamiltonian that we cast as $\mathcal{H} = \frac{1}{2} \hat{R}^T H \hat{R}$ with $\hat{R} = (\hat{R}_X, \hat{R}_Y)$ and

$$H = \begin{pmatrix} H_X & C/\sqrt{dt} \\ C^T/\sqrt{dt} & H_Y \end{pmatrix}. \quad (20)$$

Here H_X and H_Y are the individual Hamiltonians of system and ancilla and C is the $N_X \times N_Y$ matrix accounting for the interaction between them. The scaling by \sqrt{dt} is placed for convenience, as this yields simpler expressions in the limit of small dt [41].

Gaussian states are completely characterized by the first moments $r = \langle \hat{R} \rangle$ and the covariance matrix $\sigma_{ij} = \frac{1}{2} \langle \{\hat{R}_i, \hat{R}_j\} \rangle - \langle \hat{R}_i \rangle \langle \hat{R}_j \rangle$. The ancillae are assumed to be prepared in Gaussian states with zero mean, $r_Y = 0$, and generic covariance matrix σ_Y . The system, on the other hand, is prepared with arbitrary r_{X_0} and σ_{X_0} .

By compounding different infinitesimal collisions, one can construct a continuous-time dynamics for the system [56–59]. In Appendix B we provide full details on this derivation, while here we only focus on the results. The unconditional dynamics is characterized by the matrices

$$C_X = \Omega_X C, \quad C_Y = \Omega_Y C^T, \quad (21)$$

which, in general, are rectangular, with dimensions $N_X \times N_Y$ and $N_Y \times N_X$ respectively. Here Ω_X and Ω_Y are the symplectic forms with dimensions N_X and N_Y . From C_X and C_Y we then define the drift and diffusion matrices

$$A = \Omega_X H_X + \frac{1}{2} C_X C_Y, \quad D = C_X \sigma_Y C_X^T. \quad (22)$$

Examples of typical system-ancilla interactions C , as well as the resulting shapes of C_X , C_Y , A and D , are provided in Appendix C. In the continuous time limit, one then finds that the first and second moments evolve according to the following linear equations in r_X and σ_X

$$\dot{r}_X = A r_X, \quad \dot{\sigma}_X = A \sigma_X + \sigma_X A^T + D, \quad (23)$$

which we refer to as a *Lyapunov problem*.

The conditional dynamics, on the other hand, depends on two additional ingredients. The first is the functional of the covariance matrix

$$B[\sigma_X] = \sigma_X C_Y^T + C_X \sigma_Y. \quad (24)$$

As discussed in Appendix B, such functional encompasses the correlations developed between system and ancilla as a result of each collision. It is thus directly related to the information passed from the system to the ancillae. The second ingredient is the type of measurement performed on the state of the ancillae. We use here the framework of the so-called ‘‘general-dyne’’ measurements [62], whose outcomes are described by a random vector z distributed according to a multivariate Gaussian with average given precisely by the final position of the ancillae $r_{Y'} = C_Y r_X \sqrt{dt}$ [cf. Appendix B]. Such outcomes are thus directly proportional to the position of the system, but ‘‘filtered’’ by matrix C_Y . Moreover, the covariance matrix of the outcomes z is $\sigma_Y + \sigma_m$ with σ_m the covariance matrix of the noise induced by the specific choice of measurement. For a single-mode ancillary system with $\hat{R}_Y = (Q, P)$, a possible parameterization of such noise is [20, 59, 62]

$$\sigma_m = \mathcal{R}[\varphi]^T \begin{pmatrix} s/2 & 0 \\ 0 & 1/(2s) \end{pmatrix} \mathcal{R}[\varphi] + \left(\frac{1-\eta}{\eta} + \Delta \right) I/2. \quad (25)$$

The parameter $\eta \in [0, 1]$ accounts for the detector efficiency, with $\eta = 1$ describing a perfectly efficient detector and $\eta = 0$ an inefficient one. Analogously, $\Delta \in [0, \infty)$ accounts for an additive Gaussian noise, while $s \in [0, \infty)$ defines the type of measurement being used: $s = 0$ and $s = \infty$ correspond to homodyning Q and P , respectively, while $s = 1$ is for a heterodyne measurement. Finally, $\mathcal{R}[\varphi]$ is a 2×2 rotation matrix, which allows us to describe general-dyne measurement on quadratures other than Q and P .

With these ingredients, we can now completely specify the conditional dynamics by defining the matrices

$$\Lambda = C_Y^T (\sigma_Y + \sigma_m)^{-1/2}, \quad \Gamma = C_X \sigma_Y (\sigma_Y + \sigma_m)^{-1/2}, \quad (26)$$

and functional

$$\chi[\sigma] = B[\sigma] (\sigma_Y + \sigma_m)^{-1} B[\sigma]^T = (\sigma \Lambda + \Gamma) (\sigma \Lambda + \Gamma)^T. \quad (27)$$

The conditional first and second moments will then evolve according to the *Riccati problem* (stochastic)

$$\begin{aligned} dr_{X|\zeta} &= A r_{X|\zeta} dt + (\sigma_{X|\zeta} \Lambda + \Gamma) dw_t, \\ \dot{\sigma}_{X|\zeta} &= A \sigma_{X|\zeta} + \sigma_{X|\zeta} A^T + D - \chi[\sigma_{X|\zeta}], \end{aligned} \quad (28)$$

where dw_t is a vector of independent Wiener increments satisfying $\langle dw \rangle = 0$ and $\langle dw dw^T \rangle = I_Y dt$. Eqs. (28) shows that the first moments follow a dynamics induced by a stochastic Langevin equation, while the conditional covariance matrix evolves fully deterministically, thus implying that $\sigma_{X|\zeta}$ depends only on whether or not the measurement occurred and its nature, but not on the outcome ζ . This peculiarity of Gaussian systems is responsible for a significant simplification in the formal description of the process, as it will soon be illustrated.

The quantity $\chi[\sigma]$ in Eq. (27) is often referred to as the *innovation matrix* and represents the change in information from the measurement outcomes (recall that B is associated with the system-ancilla correlations). For instance, if $\eta \rightarrow 0$ in Eq. (25), the matrix σ_m diverges and hence $\chi \rightarrow 0$. The last two terms in the equation for $\sigma_{X|\zeta}$ in Eq. (28) thus represent a competition between the noise, accounted for by D , which tends to increase the modulus of the entries of $\sigma_{X|\zeta}$, and the innovation χ , which has the opposite effect.

B. Information-theoretic and thermodynamic quantities

The von Neumann entropy of an N -mode Gaussian system with covariance matrix σ and positive symplectic eigenvalues $\{v_j\}$ is given by

$$S_{\text{vN}}(\sigma) = \sum_{j=1}^N \left\{ \frac{v_j + 1}{2} \ln \frac{v_j + 1}{2} - \frac{v_j - 1}{2} \ln \frac{v_j - 1}{2} \right\}. \quad (29)$$

Using the von Neumann entropy in the Gaussian case turns out to not be always very convenient, most remarkably because of the so-called *ultra-cold catastrophe* [63], i.e. the divergence of thermodynamic quantities – such as entropy production – defined through S_{vN} that is observed when the system of interest is affected by an environment prepared in a pure state. The reasons for such divergences can be traced back to the fact that the relative entropy $D(\rho' \parallel \rho)$, which enters in the entropy production (13), diverges when the support of ρ' is not contained in the support of ρ . Yet, such a situation is very common in quantum optical experiments where the ancillae entailed by our model would be embodied by the electromagnetic field of optical modes, which is *de facto* in its vacuum state [?].

An alternative formulation for Gaussian systems is to use the Shannon entropy of the associated Wigner function [60]. Such quantity, which for Gaussian states it turns out this coincides with the Rényi-2 entropy, provides a semiclassical description in terms of phase space. Besides taking the particularly elegant form [65]

$$S_2(\sigma) = \frac{1}{2} \ln |\sigma| + N \ln 2 \quad (30)$$

with $|\sigma|$ the determinant of the covariance matrix, the Wigner entropy is not affected by divergences, even when the environmental state is pure (e.g., a $T = 0$ vacuum state modelling the interaction with an optical bath) and converges to the von Neumann entropy in the classical limit of high temperatures.

Crucially, while, in general, the Wigner entropy does not enjoy a clear information-theoretic interpretation, its Gaussian version satisfies the strong-subadditivity inequality [65, 66], a key property for an entropy to acquire an information-theoretic sense, which legitimates our choice of entropic quantifier. Without affecting the generality of our conclusions, in what follows we will omit the constant offset $N \ln 2$ from the definition of $S_2(\sigma)$.

Eq. (30) provides a form for both $S(X_t)$ and $S(X_t | \zeta_t)$, the calculation of the latter being considerably simplified by the

deterministic nature of the evolution of $\sigma_{X_t|\zeta_t}$ and its independence of ζ_t . The Holevo information Eq. (7) takes the particularly simple form

$$I(X_t; \zeta_t) = \frac{1}{2} \ln \frac{|\sigma_{X_t}|}{|\sigma_{X_t|\zeta_t}|}, \quad (31)$$

which – through the identity $\frac{d}{dt} \ln |\sigma| = \frac{1}{2} \text{tr}(\sigma^{-1} \frac{d\sigma}{dt})$, and the Riccati problem in Eqs. (28) – allows for the evaluation of the time-continuous information rate \dot{I} . We find

$$\dot{I} = \frac{1}{2} \text{tr} \{ \sigma_{X_t|\zeta}^{-1} \chi[\sigma_{X_t|\zeta}] \} - \frac{1}{2} \text{tr} \{ (\sigma_{X_t|\zeta}^{-1} - \sigma_X^{-1}) D \}, \quad (32)$$

which should be split into a gain rate \dot{G} and a loss one \dot{L} , as in Eq. (10). A detailed derivation of such splitting is presented in Appendix D, which shows that

$$\dot{G} = \frac{1}{2} \text{tr} \{ \sigma_{X_t|\zeta}^{-1} \chi[\sigma_{X_t|\zeta}] \}, \quad (33)$$

$$\dot{L} = \frac{1}{2} \text{tr} \{ (\sigma_{X_t|\zeta}^{-1} - \sigma_X^{-1}) D \}. \quad (34)$$

These results are intuitive as they demonstrate that \dot{G} is associated with the innovation matrix χ , while \dot{L} depends on the noise encoded in D . Eqs. (32)-(34) summarize the entire information dynamics in the Gaussian case.

We would like to conclude by remarking that the above calculations could also in principle be done using the von Neumann entropy in Eq. (29). We show in Fig. 2(f) that the results gathered through the Wigner and von Neumann entropy – in the context of a specific example – are nearly indistinguishable. However, the formal results obtained through the use of the von Neumann entropy are quite cumbersome, as they involve series expansions of the symplectic eigenvalues, which is a non-trivial task. Using the Wigner entropy therefore offers a significant simplification, making the interpretation of the results much clearer.

C. Conditions for the establishment of ISS

Eq. (33) provides a clear condition for the existence of an ISS. Recalling the definition in Eq. (27), we can write \dot{G} in the more symmetric form

$$\dot{G} = \frac{1}{2} \text{tr} \{ \sigma_{X_t|\zeta}^{-1/2} B[\sigma_{X_t|\zeta}] (\sigma_Y + \sigma_m)^{-1} B[\sigma_{X_t|\zeta}]^T \sigma_{X_t|\zeta}^{-1/2} \}. \quad (35)$$

This is clearly the trace of a positive semi-definite matrix and both $\sigma_{X_t|\zeta}$ and $\sigma_Y + \sigma_m$ are quantum covariance matrices (which are therefore always strictly positive definite). Thus,

$$\dot{G} = 0, \quad \text{iff} \quad B[\sigma_{X_t|\zeta}] = 0. \quad (36)$$

Of course, this assumes that the entries of the noise matrix σ_m are finite; that is, that the measurement is not completely uninformative. For instance, if $\eta \rightarrow 0$ and/or $\Delta \rightarrow \infty$ in Eq. (25), clearly we would have $\dot{G} = 0$ even if $B \neq 0$.

D. Thermodynamic analysis

Next we turn to the thermodynamics of the system. First, we evaluate the entropy flux, which in continuous-time takes the form of a rate

$$\dot{\Phi} = \text{tr} \{ A \} + \frac{1}{2} \text{tr} \{ \sigma_Y^{-1} \tilde{\sigma}_X \} + \frac{1}{2} \tilde{r}_X^T \sigma_Y^{-1} \tilde{r}_X, \quad (37)$$

with $\tilde{r}_X = C_Y r_X$ and $\tilde{\sigma}_X = C_Y \sigma_X C_Y^T$. Similarly, the unconditional entropy production rate in Eq. (12) becomes

$$\begin{aligned} \dot{\Sigma}_u &= 2 \text{tr} \{ A \} + \frac{1}{2} \text{tr} \{ \sigma_X^{-1} D + \sigma_Y^{-1} \tilde{\sigma}_X \} + \frac{1}{2} \tilde{r}_X^T \sigma_Y^{-1} \tilde{r}_X \\ &= \dot{\Phi} + \text{tr} \{ A \} + \frac{1}{2} \text{tr} \{ \sigma_X^{-1} D \}. \end{aligned} \quad (38)$$

Finally, we can compute the conditional entropy production $\dot{\Sigma}_c$ using Eq (17)

$$\begin{aligned} \dot{\Sigma}_c &= 2 \text{tr} \{ A \} + \frac{1}{2} \text{tr} \{ \sigma_Y^{-1} \tilde{\sigma}_X + \sigma_{X_t|\zeta}^{-1} (D - \chi[\sigma_{X_t|\zeta}]) \} + \frac{1}{2} \tilde{r}_X^T \sigma_Y^{-1} \tilde{r}_X \\ &= \dot{\Phi} + \text{tr} \{ A \} + \frac{1}{2} \text{tr} \{ \sigma_{X_t|\zeta}^{-1} (D - \chi[\sigma_{X_t|\zeta}]) \}. \end{aligned} \quad (39)$$

The last line in Eq. (39) shows clearly that $\dot{\Sigma}_c$ coincides with the expression of $\dot{\Sigma}_u$ where $\sigma_X \rightarrow \sigma_{X_t|\zeta}$ and $D \rightarrow D - \chi[\sigma_{X_t|\zeta}]$, but without changing the associated entropy flux rate. This, together with Eqs. (37) and (38), completely summarize the thermodynamics of continuous variable CM²s.

IV. EXAMPLES AND APPLICATIONS

In this Section we illustrate the potential of the framework developed so far by tackling a paradigmatic example and then moving to the modelling of an experiment in an optomechanical platform akin to the situation recently reported in Ref. [7].

A. Example: two-mode ancilla

We analyze a two-mode ancilla problem [cf. Fig. 2(a)], where the first ancilla is prepared in its vacuum state while the second in a squeezed state of squeezing degree ξ . The covariance matrix of the environmental state is thus

$$\sigma_Y = \begin{pmatrix} \mathbb{I}/2 & \mathbb{O} \\ \mathbb{O} & \mathbb{S}/2 \end{pmatrix} \quad (40)$$

with \mathbb{I} (\mathbb{O}) the 2×2 identity (null) matrix and $\mathbb{S} = \text{diag}(e^{2\xi}, e^{-2\xi})$. The interest of this choice lies also on the fact that both ancillary sub-systems are here prepared in a pure state, which makes the standard formulation based on the von Neumann entropy inapplicable in light of the ultra-cold catastrophe. We assume the system interacts sequentially with each ancilla. Moreover, we take the interactions to be of an excitation-exchange type (cf. Appendix (C 1) for details),

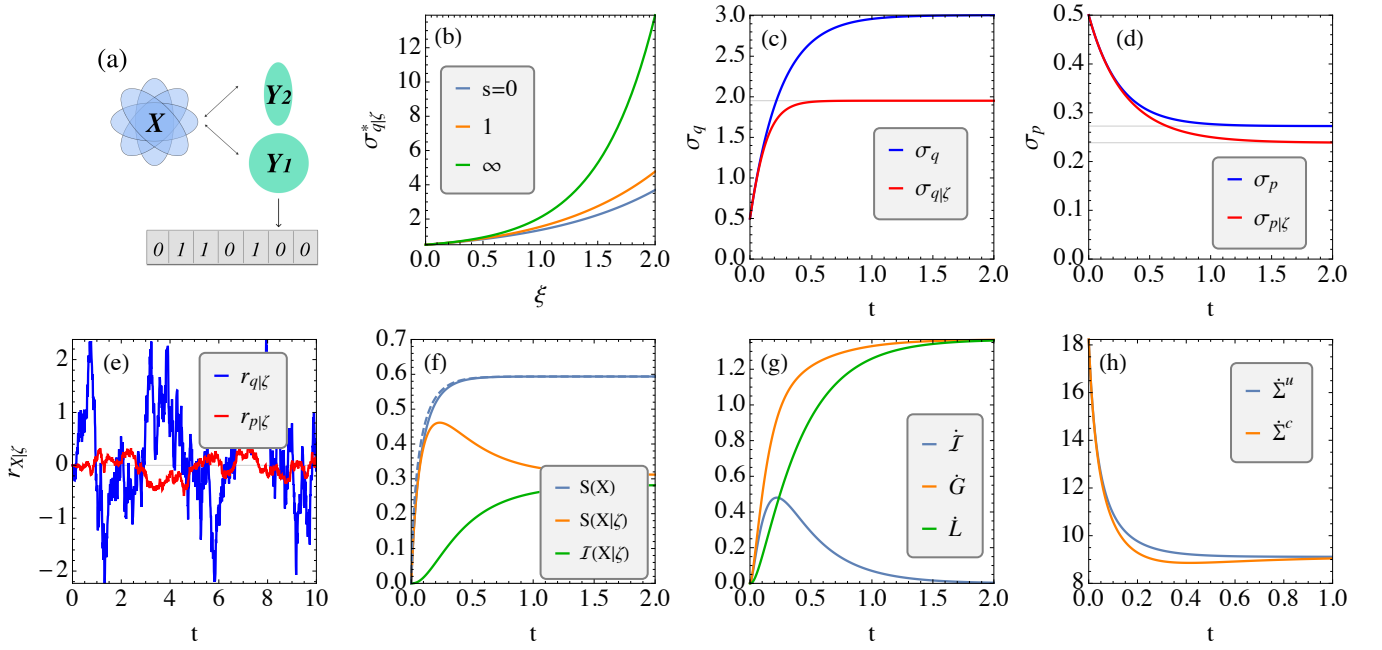


FIG. 2. (a) CM^2 in the continuous-variable scenario, with a two-mode ancilla prepared in the state (40) and interacting with the beam-splitter matrix (41), with $\gamma = 1$. (b) Conditional steady-state variance of the q quadrature for different measurement strategies, as a function of the squeezing ξ in the ancilla state: $s = 0$, homodyne in Q , $s = 1$, heterodyne and $s = \infty$ homodyne in P . For the latter, σ_q^* coincides with the unconditional variance [Eq. (43)]. (c), (d) Dynamics of the conditional and unconditional variances of q and p , assuming the system starts in the vacuum, with $s = 1$ and $\xi = 1.2$. (e) Sample trajectories of the conditional first moments. The choice of squeezing $\xi = 1.5$ in the ancilla cause $r_{q|\xi}$ to fluctuate significantly more than $r_{p|\xi}$. (f) Unconditional and conditional entropies, and Holevo information (7). The dashed line superimposed on the blue curve is the unconditional von Neumann entropy, scaled by a constant factor. For Gaussian states, it is nearly indistinguishable from the Rényi-2. (g) Information rate, \dot{I} , computed from Eq. (32), together with the information gain and loss rates \dot{G} and \dot{L} [Eqs. (33) and (34)]. The model presents an information steady-state, evidenced by a finite \dot{G} even in the long-time limit. (h) Unconditional and conditional entropy production rates computed from Eqs. (38) and (39).

which results in a partial SWAP of the states of the colliding systems. The interaction matrix C in Eq. (20) takes the form

$$C = \sqrt{2\gamma} \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad (41)$$

where γ is the interaction strength. Finally, we assume only the first ancilla is measured. That is, we choose the measurement matrix σ_m to be of the form [c.f. Eq. (25)]

$$\sigma_m = \frac{1}{2} \text{diag} \left(s, \frac{1}{s}, \frac{1-\eta}{\eta}, \frac{1-\eta}{\eta} \right). \quad (42)$$

We then eliminate the information on the second ancilla by taking $\eta \rightarrow 0$.

The unconditional steady-state is readily found by setting $\dot{\sigma}_X = 0$ in Eq. (23), which gives

$$\sigma_X^* = \frac{\mathbb{S} + \mathbb{I}}{4} \quad (43)$$

This is the average between the initial states of the two ancillae: The alternating collisions cause the system to homogenize to a state that is just the mean between the two states.

Similarly, we can also compute the conditional steady-state, by solving the equation $\dot{\sigma}_{X|\xi} = 0$ in the Riccati problem of

Eq. (28). The result is

$$\sigma_{X|\xi}^* = \frac{1}{2} \begin{pmatrix} \sqrt{(1+s)(2e^{2\xi} + s)} - s & 0 \\ 0 & \frac{\sqrt{(1+s)(2e^{-2\xi} + s)} - 1}{s} \end{pmatrix}. \quad (44)$$

The variance $(\sigma_{X|\xi}^*)_{11}$ is shown in Fig. 2(b) against ξ and for different measurement choices s . For $s \rightarrow \infty$, this tends to the unconditional value $(\sigma_q^*)_{11} = (e^{2\xi} + 1)/4$, while for $s = 0$ it gives $e^\xi/2$. For any value of s , we always have $\sigma_{q|\xi}^* \leq \sigma_q^*$. Measuring therefore *always* cools down both quadratures. However, the cooling performance depends on the type of measurement being considered.

Figs. 2(c) and (d) shows the dynamics of the elements of the covariance matrix of X , which is assumed to be initially prepared in the vacuum state. Similarly, Fig. 2(e) presents sample trajectories of the conditional first moments, $r_{q|\xi}$ and $r_{p|\xi}$. The figure shows the results gathered by taking $s = 1$, which corresponds the performance of heterodyne measurement, so that the measurement is symmetric in both quadratures. However, the behavior of the q and p quadratures is fundamentally different. This is a consequence of the choice of initial ancilla state. We have chosen $\xi = 1.2$, meaning that Y_2 is squeezed in the P direction (and hence expanded in the Q direction). As a consequence, the steady-state covariance σ_q

is much larger than that of p , for both the conditional and unconditional dynamics. Interestingly, though, we also see that the cooling effect of measurement is much more significant in the q quadrature.

The results for the variances are reflected on both the information and thermodynamics of this example. In Fig 2(f) we plot the unconditional and conditional entropies, as well as the Holevo information. The conditional entropy tends to a lower value than the unconditional one, in agreement with the findings for the variances. This is done by acquiring information. Fig. 2(g) shows the information rate \dot{I} , computed from Eq. (32). Initially a lot of information is acquired, but as time passes \dot{I} tends to zero. However, the gain rate and loss rates, Eqs. (33) and (34), tend to a finite value in the steady-state, thus characterizing an ISS. Finally, a comparison between the unconditional and conditional evolution of the entropy production is shown in Fig. 2(h), where it can be seen that both decay from an initially high value towards a non-zero steady-state value. The reason why the entropy production rate is initially high is because the initial state of the system is very far from equilibrium. Also, the difference $\sigma^c - \sigma^u$ is larger for intermediate times, which is when \dot{I} is largest. At $t = 0$ and at $t = \infty$, both quantities coincide, as they should.

The results of Fig. 2 clearly show that the system tends to an ISS. According to Eq. (36), the condition for this to be the case is to have $B[\sigma_{X|q}^*] > 0$. In our case, using Eqs. (40), (41) and (44), we find

$$B[\sigma_{X|q}^*] = \sqrt{\frac{\gamma}{2}} \begin{pmatrix} 1 - 2\sigma_{Q|q}^* & 0 & e^{2\xi} - 2\sigma_{Q|q}^* & 0 \\ 0 & 1 - 2\sigma_{P|q}^* & 0 & e^{-2\xi} - 2\sigma_{P|q}^* \end{pmatrix}$$

where $\sigma_{Q|q}^*$ and $\sigma_{P|q}^*$ are the entries of Eq. (44). We therefore see the conditions for the existence of an ISS are quite light. Essentially, as long as the steady-state of the system is neither that of Y_1 nor that of Y_2 , information will continue to be acquired in every collision. This result also provides guidelines on how different measurement strategies affect the ISS. For instance, suppose we were to measure ancilla Y_2 instead of Y_1 . From Eq. (35) we have that $\dot{G} \propto B(\sigma_Y + \sigma_m)^{-1} B^T$ and measuring Y_2 means introducing an infinite amount of noise in the Y_1 block of σ_m . This would then eliminate the left block of B .

B. Global vs. reduced dynamics

We now use the continuous variable results to make a small digression about an important point in quantum and stochastic thermodynamics. Consider a general scenario of a system interacting with a bath. Very often, this process is described by an effective reduced description, such as a quantum master equation. The point we wish to address is that, while this description may be adequate for describing the dynamics, it does not necessarily suffice to describe the thermodynamics [28]. This can be illustrated by the following minimal example. Consider a qubit subject to a standard thermal bath, as described by the Gorini-Kossakowski-Lindblad-

Sudarshan (GKLS) master equation

$$\begin{aligned} \frac{d\rho}{dt} &= -i[H, \rho] + \mathcal{D}(\rho) \\ &= -i[H, \rho] + \gamma(\bar{n} + 1)\mathcal{D}_- + \gamma\bar{n}\mathcal{D}_+, \end{aligned} \quad (45)$$

where $H = \omega\sigma_z/2$, $\mathcal{D}_\mp = \sigma_\mp\rho\sigma_\pm - \frac{1}{2}\{\sigma_\pm\sigma_\mp, \rho\}$ with $\sigma_- = \sigma_+^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\gamma > 0$ is the dissipation rate, and $\bar{n} = (e^{\omega/T} - 1)^{-1}$ is the mean number of excitations in the bath. In this scenario, one would naturally associate the heat current to the bath with

$$\frac{d\langle H \rangle}{dt} = \text{tr}\{H\mathcal{D}(\rho)\}. \quad (46)$$

In the long-time limit the system will tend to an equilibrium state, characterized by a zero current. However, suppose instead that the same system is coupled to two baths, at different temperatures T_1 and T_2 . In a weak-coupling approximation, the resulting master equation for the two baths will be additive, so that we would simply have

$$\frac{d\rho}{dt} = -i[H, \rho] + \mathcal{D}_1(\rho) + \mathcal{D}_2(\rho), \quad (47)$$

where each dissipator \mathcal{D}_i is defined as in Eq. (45), with parameters γ_i and \bar{n}_i . Eq. (47) can be recast in a form involving a single dissipator by defining $\mathcal{D}' = \mathcal{D}_1 + \mathcal{D}_2$, which would be of the form of Eq. (45) with parameters $\gamma' = \gamma_1 + \gamma_2$ and $\bar{n}' = (\gamma_1\bar{n}_1 + \gamma_2\bar{n}_2)/(\gamma_1 + \gamma_2)$. While such reformulation would suggest thermalization with a single bath at the effective temperature $T' = \omega/\ln(1 + 1/\bar{n}')$, in reality, the process itself is clearly different and would drive the system to a non-equilibrium steady state. This is seen from the fact that, as long as $T_1 \neq T_2$, we will have $\text{tr}\{H\mathcal{D}_i\} \neq 0$, meaning there will be a current of heat from one bath to the other. However, this can only be observed if one has access to the additional information that $\mathcal{D}' = \mathcal{D}_1 + \mathcal{D}_2$ and the local currents. A rigorous thermodynamic description therefore requires that one properly identifies all possible heat sources. Despite its simplicity, this example illustrates well how the thermodynamic interpretation can be *fundamentally* altered depending on the amount of global information one has access to. Note also that this is not a quantum feature, as the same problem also appears in stochastic thermodynamics, as discussed in detail, e.g. in Ref. [67].

We will now analyze this issue from the viewpoint of continuous variable models. The reduced dynamics of the system is specified by the four matrices A, D, Γ and Λ , while the full global dynamics is also specified by σ_Y, C_X and C_Y . Any property that can be expressed solely as a function of the former set of matrices can thus be found only from the reduced dynamics. Let us then analyze the entropy flux rate Eq. (37) from this perspective. We can rewrite the flux rate in terms of $\Upsilon = C_Y^T \sigma_Y^{-1} C_Y$ as

$$\dot{\Phi} = \text{tr}\{A\} + \frac{1}{2} \text{tr}\{\Upsilon\sigma_X\} + \frac{1}{2} r_X^T \Upsilon r_X.$$

In general Υ cannot be constructed solely from A, D, Γ, Λ , which shows the global character of the flux. This becomes

quite important in light of the additivity property of the flux rate stated in Eq. (14). To see this, suppose Y has multiple – initially independent – internal units, so that $\sigma_Y = \bigotimes_j \sigma_{Y_j}$. This entails

$$C_X = (C_{X_1} \ C_{X_2} \ \dots), \quad C_Y = \begin{pmatrix} C_{Y_1} \\ C_{Y_2} \\ \vdots \end{pmatrix}, \quad (48)$$

and as a consequence $\Upsilon = \sum_j C_{Y_j}^T \sigma_{Y_j}^{-1} C_{Y_j} = \sum_j \Upsilon_j$. We also define the dissipative part of matrix A as $A_d = \frac{1}{2} C_X C_Y \sum_j A_{dj}$ (with $A_{dj} = \frac{1}{2} C_{X_j} C_{Y_j}$), which is the only part contributing to $\text{tr}\{A\}$. The flux rate is thus additive, and reads

$$\dot{\Phi} = \sum_j \Phi_j = \frac{1}{2} \sum_j \left[\text{tr}\{C_{X_j} C_{Y_j}\} + \frac{1}{2} \text{tr}\{\Upsilon_j \sigma_X\} + \frac{1}{2} r_X^T \Upsilon_j r_X \right]. \quad (49)$$

Each term in the sum identifies the entropy flux rate to the individual ancillae and, hence, to each independent source of dissipation within the system. In the particular case where the matrices C_{X_j} are invertible, we can also relate Υ with the diffusion matrix D [cf. Eq. (22)], thus giving the alternative decomposition of the flux rate as

$$\dot{\Phi} = \sum_j \left[\text{tr}\{A_{dj}\} + 2 \text{tr}\{A_{dj}^T D_j^{-1} A_{dj} \sigma_X\} + 2 r_X^T A_{dj}^T D_j^{-1} A_{dj} r_X \right]. \quad (50)$$

This expression preserves the correct identification of the dissipation channels. Note, however, that it requires not only A_d and D , but also their specific decompositions in terms of A_{dj} and D_j . Finally, we also mention that the flux rate can only be expressed in the form in Eq. (50) if the matrices C_{X_j} are invertible. There are many cases when this does not hold true, as illustrated for instance in Appendix C 2. In those cases, one must rely on the original expression Eq. (49), which holds for any interaction matrix.

C. Modeling an optomechanical experiment

Finally, we employ our framework to describe the experiment performed in Ref. [7]. The setup consists of an intracavity mechanical mode embodied by a vibrating membrane, subjected to two external baths. The first is a standard thermal bath, associated with a phononic background for the mechanical mode. The second bath is optical, and provided by the field of the cavity, which is eliminated adiabatically from the dynamics of the system and, by being continuously monitored, gives information on the mechanical system. The scenario is therefore similar in spirit to the two-mode example of Fig. 2.

The evolution of the system is described by the stochastic master equation [68, 69]

$$d\rho = \mathcal{L}_{\text{th-qba}} dt + \sqrt{\eta \Gamma_{\text{qba}}} \left(\mathcal{H}[q] dw_1 + \mathcal{H}[p] dw_2 \right), \quad (51)$$

where

$$\mathcal{L}_{\text{th}} = (\Gamma_m(\bar{n} + 1) + \Gamma_{\text{qba}}) \mathcal{L}_- + (\Gamma_m \bar{n} + \Gamma_{\text{qba}}) \mathcal{L}_+, \quad (52)$$

represents the Lindblad dissipator, including the quantum back-action mechanism induced by the monitored optical baths, with \mathcal{L}_{\pm} being analogous to \mathcal{D}_{\pm} with $\sigma_- \rightarrow a$, $\sigma_+ \rightarrow a^\dagger$, and (a, a^\dagger) the bosonic operators of the mechanical mode. In Eq. (51) we also defined $\mathcal{H}[O] = O\rho + \rho O^\dagger - \rho \text{tr}((O + O^\dagger)\rho)$, which is associated with the continuous measurements, with $\eta \in [0, 1]$ denoting the measurement efficiency. Finally, dw_1 and dw_2 are two independent Wiener increments.

Eq. (51) leads to unconditional and conditional dynamics described by Eqs. (23) and (28), with

$$A = -\frac{\Gamma_m}{2} \mathbb{I}, \quad D = \left[\Gamma_m \left(\bar{n} + \frac{1}{2} \right) + \Gamma_{\text{qba}} \right] \mathbb{I}, \quad \chi[\sigma] = 4\eta \Gamma_{\text{qba}} \sigma_X^2. \quad (53)$$

These results are intuitive: the diffusion matrix D , which is responsible for the noise, is associated to both the thermal and optical baths. The innovation matrix χ , on the other hand, is associated only to the optical bath, which is the only one being measured. Moreover, $\chi \propto \eta$, so that if $\eta = 0$ (fully inefficient measurement), the innovation vanishes. Slightly less intuitive is the fact that the optical bath does not affect the damping matrix A . This is associated to the specific way in which the optical mode couples to the mechanical system.

In order to be able to properly describe the thermodynamics of this system, we must now construct a CM² which reproduces the matrices in Eqs. (53) at the level of the reduced dynamics, looking for its minimal construction.

First, it is worth stressing that the thermal and optical baths are independent. The former does not have to be modelled by a collisional model as it is not monitored. Its effects could thus be described by a master equation. In order to better match with the notation of the remainder of the paper, however, we shall assume that the thermal part is described by a collisional model as well. In this case, it can be generated by using a single-mode thermal ancilla interacting with the system via an excitation-exchange interaction, as studied in Sec. IV A. The description of the optical mode, on the other hand, is less trivial. A special feature of the stochastic master equation Eq. (51) is that it allows one to independently monitor the two mechanical quadratures q and p . The optical bath must, therefore, be itself composed of at least two modes. Thus, the ancilla in this model must have a total of 3 modes, with initial state

$$\sigma_Y = \begin{pmatrix} (\bar{n} + 1/2) \mathbb{I} & \circ & \circ \\ \circ & \mathbb{I}/2 & \circ \\ \circ & \circ & \mathbb{I}/2 \end{pmatrix}. \quad (54)$$

The first ancillary mode is thus in a thermal state with occupation number \bar{n} , while the optical ancillae are initially in the vacuum state.

Next we turn to the interaction matrices for the optical baths. The peculiar feature of this interaction is that it generates no contribution to the damping matrix A in Eq. (53). As discussed in Appendix C 2, this feature is generated by position-position or momentum-momentum couplings: the second ancilla (i.e. the first optical one) is used to monitor q via the interaction Hamiltonian $-\sqrt{2\Gamma_{\text{qba}}} q Q_2$, while the third ancilla monitors the mechanical momentum p via the term

$\sqrt{2\Gamma_{\text{qba}}}pP_3$. The interaction matrix C will thus have the form

$$C = \begin{pmatrix} 0 & -\sqrt{\Gamma_m} & -\sqrt{2\Gamma_{\text{qba}}} & 0 & 0 & 0 \\ \sqrt{\Gamma_m} & 0 & 0 & 0 & 0 & \sqrt{2\Gamma_{\text{qba}}} \end{pmatrix}. \quad (55)$$

This then leads to the matrices [cf. Eq. (21)]

$$C_X = \left(\sqrt{\Gamma_m} \mathbb{I} \quad \sqrt{2\Gamma_{\text{qba}}} \sigma_- \quad \sqrt{2\Gamma_{\text{qba}}} \sigma_- \right), \quad (56)$$

$$C_Y = \begin{pmatrix} -\sqrt{\Gamma_m} \mathbb{I} \\ \sqrt{2\Gamma_{\text{qba}}} \sigma_- \\ \sqrt{2\Gamma_{\text{qba}}} \sigma_+ \end{pmatrix}. \quad (57)$$

The drift and diffusion matrices Eqs. (22) will then be given by Eq. (53).

Finally, to reproduce the innovation matrix in Eq. (53), the measurement matrix σ_m must have the form $\sigma_m = \bigoplus_{j=1}^3 \sigma_{m_j}$, where each σ_{m_j} is given by Eq. (25), with the following parameters: First, for the thermal ancilla $\eta_1 = 0$ with s_1 arbitrary. Then, while the first optical ancilla is coupled through a position-position mechanism to the mechanical system, we actually have to detect its momentum P_2 . That is, we set $\eta_2 = \eta$ and $s_2 = \infty$. Finally, for the second optical ancilla, we set $\eta_3 = \eta$ and $s_3 = 0$, so that we homodyne Q_3 . With these choices for the matrix σ_m , one then reproduces exactly the innovation matrix in Eq. (53). The minimal CM² that we have deduced here can consistently describe the dynamics and thermodynamics of the experiment performed in Refs. [7, 27].

In particular, in Ref. [27], the analysis of the thermodynamics of the system addressed here was presented by working at the level of the system master equation. However, in order to obtain the correct expression of the entropy fluxes, a refined analysis was needed, starting from the full mechanical-cavity system master equation before the adiabatic elimination of the latter could be performed [68, 69] leading to Eq. (51). Here we can clearly see why that was the case. Indeed, we can see from Eq. (56) that the matrices C_{X_i} for $i = 2, 3$ — which reproduce the dynamics as described by Eq. (51) obtained after adiabatic elimination and other approximations — are singular. Following our previous discussion, the entropy flux in this case cannot be written in terms of the sole reduced dynamics. This strengthens further the point we made in the previous Section on the importance of global information for the consistent description of the thermodynamics of the system.

V. CONCLUSIONS

We have investigated the interplay between information and thermodynamics in continuously measured system by way of a collisional model construct. In particular, we were able to formulate the entropy production and flux rate — two pivotal quantities in (quantum) thermodynamics — from a purely informational point of view and accounting for repeated indirect measurements of the system of interest. These results offer a clear way to point-out and characterise the effect of quantum

measurements on the thermodynamics of open quantum system. Moreover, they generalized recent theoretical and experimental results [20, 27] beyond Gaussian systems and dynamics.

We model the indirect measurement of the system via a collisional model where (a part of) the environment with which the system interact is monitored. This allows us to compare the entropy production with the case in which the environment is not measured and the evolution of the system is thus unconditioned. In turn, this comparison leads directly to a tightened second law for monitored systems with a very clear separation between entropic contributions coming from the dissipative interaction with the environment and the ones coming from the information gained during the monitoring. This neat separation also allows us to introduce the concept of information gain rate and loss rates, and informational steady-state. The latter are particularly interesting since they represent cases where a delicate balance is established between the information that gets lost into the environment and the one that is extracted by measuring.

The formalism developed in this work is widely applicable as exemplified by the case study considered, from qubit models to continuous variable systems. In fact, we also provide an account of a recent experiment in optomechanics, showing that our formalism should be of useful in describing a broad variety of quantum-coherent experiments.

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Appendix A: Classical (incoherent) CM²

It is interesting to enquire what are the classical analogs of the quantum put forth in Sec. II. Or, put it differently, what are the conditions for the model to be called classical, or incoherent.

Let us focus on a single collision event. We assume that, at a certain instant of time, the system is at $\rho_X = \sum_x p(x)|x\rangle\langle x|$ for some basis $|x\rangle$, while the ancilla is prepared in $\rho_Y = \sum_y p(y)|y\rangle\langle y|$, for some basis $|y\rangle$. The unconditional state of the system after one collision will then be

$$\rho'_X = \mathcal{E}(\rho_X) = \sum_{x,y,y'} p(x)p(y)\langle y'|U|xy\rangle\langle xy|U^\dagger|y'\rangle,$$

where $\langle y'|U|xy\rangle$ is still a ket in the Hilbert space of the system. This ket is not normalized, however, so we define

$$|\Psi_{xyy'}\rangle := \frac{\langle y'|U|xy\rangle}{\sqrt{P(y'|xy)}}, \quad P(y'|xy) = \|\langle y'|U|xy\rangle\|^2. \quad (A1)$$

The state of the system may then be written as

$$\rho'_X = \sum_{xyy'} p(x)p(y)P(y'|xy)|\Psi_{xyy'}\rangle\langle\Psi_{xyy'}|.$$

When written in this way, it gives the impression that ρ'_X is already in diagonal form. But this is not the case, since in general the states $|\Psi_{xyy'}\rangle$ are not orthogonal and do not form a basis. Moreover, there are usually many more states than that required to span the Hilbert space of X (there can be up to $d_X d_Y^2$ of them, where d_X, d_Y are the dimensions of system and ancilla). As a matter of fact, in general the eigenvectors of ρ'_X will have no simple relation with the states $|\Psi_{xyy'}\rangle$.

Conversely, we say a model is *unconditionally incoherent* if for any xyy' , the states $|\Psi_{xyy'}\rangle$ are always elements of the basis $|x\rangle$. In this case ρ'_X will be automatically diagonal,

$$\rho'_X = \sum_{x'} p(x')|x'\rangle\langle x'|, \quad (\text{A2})$$

where the populations $p(x')$ can be found from

$$p(x') = \langle x'|\rho'_X|x'\rangle = \sum_{x,y,y'} p(x)p(y)P(y'|xy)\langle x'|\Psi_{xyy'}\rangle\langle\Psi_{xyy'}|x'\rangle.$$

Using (A1), we can also write this as

$$p(x') = \sum_{x,y,y'} Q(x'y'|xy)p(x)p(y), \quad (\text{A3})$$

where

$$Q(x'y'|xy) = |\langle x'y'|U|xy\rangle|^2, \quad (\text{A4})$$

$$\mathcal{E}_z(\rho_X) = \sum_{xyy''y'''} p(x)p(y)\langle y'|M_z|y''\rangle\langle y''|U|xy\rangle\langle xy|U^\dagger|y'''\rangle\langle y'''|M_z^\dagger|y'\rangle.$$

If the model is unconditionally incoherent, the states $\langle y''|U|xy\rangle$ will be elements of the basis $|x\rangle$. But the resulting state will in general not be diagonal due to the terms $\langle y'|M_z|y''\rangle$ and $\langle y'''|M_z^\dagger|y'\rangle$. In other words, coherence may very well be produced by the measurement itself. And while this cannot affect the unconditional dynamics of the system (due to no-signaling), it may very well affect the conditional one.

We therefore define a model to be *conditionally incoherent* if it is unconditionally incoherent *and* if

$$\langle y'|M_z|y''\rangle \propto \Delta_{y',y''}.$$

The simplest possibility would, of course, be to take M_z as projective measurements in the basis $|y\rangle$. But there may also be other interesting possibilities. For instance, we can take M_z to be an imprecise projective measurement, which only runs

is the transition probability of observing a transition $(x, y) \rightarrow (x', y')$. A matrix of this form is said to be unistochastic, which is a particular case of doubly stochastic matrices.

An example of a unconditionally incoherent model is when both system and ancillae are qubits, interacting with the partial SWAP

$$U = (|00\rangle\langle 00| + |11\rangle\langle 11|) + \lambda(|01\rangle\langle 01| + |10\rangle\langle 10|) - i\sqrt{1-\lambda^2}(|01\rangle\langle 10| + |10\rangle\langle 01|). \quad (\text{A5})$$

In this case

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda^2 & 1-\lambda^2 & 0 \\ 0 & 1-\lambda^2 & \lambda^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (\text{A6})$$

with $\lambda \in [0, 1]$.

In unconditionally incoherent models, if the system is originally diagonal in the basis $|x\rangle$, it will remain so throughout the evolution, with the populations evolving according to the classical Markov chain

$$p(x_{t+1}) = \sum_{x_t} Q(x_{t+1}|x_t)p(x_t), \quad Q(x'|x) = \sum_{y,y'} Q(x'y'|xy)p(y). \quad (\text{A7})$$

Next we can do the same for the conditional map \mathcal{E}_z in Eq. (5). As we will see, however, unconditional incoherence does not imply conditional incoherence. Following the same steps as before, we can write

$$\mathcal{E}_z(\rho_X) = \sum_{xyy'} p(x)p(y)\langle y'|M_z U|xy\rangle\langle xy|U^\dagger M_z^\dagger|y'\rangle.$$

We now introduce two completeness relations in the y basis:

over certain elements of the basis $|y\rangle$. Or we could make M_z be a noisy measurement, that blurs the outcomes of each $|y\rangle$. It is worth noting, in passing, that conditional incoherence also immediately implies the validity of Eq. (??) on the entropy fluxes for conditionally incoherent models.

In any case, when the model is conditionally incoherent the map (5) can be written as

$$\mathcal{E}_z(\rho_X) = \sum_{x,y,y'} p(x)p(y)M(z|y')P(y'|xy)|\Psi_{xyy'}\rangle\langle\Psi_{xyy'}|, \quad (\text{A8})$$

where

$$M(z|y') = |\langle y'|M_z|y'\rangle|^2 = \text{tr}[M_z^\dagger M_z|y'\rangle\langle y'|], \quad (\text{A9})$$

is the conditional probability of observing outcome z , given that the ancilla is in $|y'\rangle$. This therefore represents the “post-

processing” of the ancillary state. The state (A8) can also be written as

$$\mathcal{E}_z(\rho_X) = \sum_{x'} p(x', z) |x'\rangle\langle x'|, \quad (\text{A10})$$

where

$$p(x', z) = \sum_{x, y, y'} p(x) p(y) M(z|y') Q(x' y' | xy).$$

This is consistent with Eq. (??): since the result of the map is a distribution in both x' and z , if we trace over X we are left only with $p(z)$.

At this point it is convenient to define the transition matrix

$$W(x' z | x) = \sum_{y, y'} M(z|y') Q(x' y' | xy) p(y). \quad (\text{A11})$$

In a classical context, this is the most important object defining a CM². It describes the (Markovian) transition probability, of observing the system in x' , as well as the outcome z , given that initially the system was in x . With this definition, it follows that

$$p(x', z) = \sum_x W(x' z | x) p(x),$$

which, classically, is precisely what one would expect from the law of total probability.

Finally, we adapt these ideas to multiple collisions. The initial state of the system is $\rho_{X_0} = \sum_{x_0} p(x_0) |x_0\rangle\langle x_0|$. The conditional (unnormalized) state after the first collision is obtained by applying (A10):

$$\varrho_{X_1 | \zeta_1} = \sum_{x_1} p(x_1, \zeta_1) |x_1\rangle\langle x_1|, \quad p(x_1, \zeta_1) = \sum_{x_0} W(x_1 \zeta_1 | x_0) p(x_0),$$

where, recall $\zeta_1 = z_1$. Similarly, after the second collision, the conditional state will be $\varrho_{X_2 | \zeta_2} = \sum_{x_2} p(x_2, \zeta_2) |x_2\rangle\langle x_2|$, where

$$p(x_2, \zeta_2) = \sum_{x_0, x_1} W(x_2 \zeta_2 | x_1) W(x_1 \zeta_1 | x_0) p(x_0),$$

Proceeding in this way, we then see that after the t -th collision, the state of the conditional system will then be

$$\varrho_{X_t | \zeta_t} = \sum_{x_t} p(x_t, \zeta_t) |x_t\rangle\langle x_t|, \quad (\text{A12})$$

where

$$p(x_t, \zeta_t) = \sum_{x_0, \dots, x_{t-1}} W(x_t \zeta_t | x_{t-1}) \dots W(x_1 \zeta_1 | x_0) p(x_0), \quad (\text{A13})$$

Tracing over this state and recalling Eq. (??), we then finally obtain the distribution of outcomes

$$P(\zeta_t) = \sum_{x_0, \dots, x_t} W(x_t \zeta_t | x_{t-1}) \dots W(x_1 \zeta_1 | x_0) p(x_0) \quad (\text{A14})$$

This result is quite important, as it clearly highlights the hidden Markov structure of the present model, discussed in Sec. II.

Summarizing, the incoherent version of a CM² is completely defined by the transition matrix $W(x' z | x)$ in Eq. (A11). This, in turn, depends on the transition matrix $Q(x' y' | xy)$ in Eq. (A4), which must be unistochastic, and the noise matrix $M(z' | y)$, which can be any conditional probability.

Appendix B: Construction of the Gaussian CM²

In this appendix we detail the derivation of the main results of Sec. III A. We begin by focusing on a single collision described by the Hamiltonian (20). The Heisenberg evolution of the quadratures after a time dt is given by $\hat{R}(dt) = e^{i\Omega_{XY} H dt} \hat{R}(0)$, where Ω_{XY} is the symplectic form of dimensions $2(N_X + N_Y)$. Expanding for small dt , we then find

$$\hat{R}_{X'} = \hat{R}_X + A \hat{R}_X dt + C_X \hat{R}_Y \sqrt{dt}, \quad (\text{B1})$$

$$\hat{R}_{Y'} = \hat{R}_Y + A_Y \hat{R}_Y dt + C_Y \hat{R}_X \sqrt{dt}, \quad (\text{B2})$$

where the matrix A is defined in Eq. (22). In addition, we also defined

$$A_Y = \Omega_Y H_Y + \frac{1}{2} C_Y C_X. \quad (\text{B3})$$

From Eqs. (B1) and (B2) we then find that the first moments, after the collision, are given by

$$r_{X'} = r_X + A r_X dt, \quad (\text{B4})$$

$$r_{Y'} = C_Y r_X \sqrt{dt}, \quad (\text{B5})$$

The ancilla is displaced by an amount proportional to r_X . But this is “filtered” by C_Y , which can cause the ancilla to become blind to some of the system’s quadratures. This becomes particularly clear from the examples discussed in Appendix C.

Similarly, we can look at the evolution of the second moments. We parametrize the covariance matrix after the collision as

$$\sigma_{X'Y'} = \begin{pmatrix} \sigma_{X'} & \xi_{X'Y'} \\ \xi_{X'Y'}^T & \sigma_{Y'} \end{pmatrix}. \quad (\text{B6})$$

Eqs. (B1) and (B2) then yield

$$\sigma_{X'} = \sigma_X + (A \sigma_X + \sigma_X A^T + D) dt, \quad (\text{B7})$$

$$\sigma_{Y'} = \sigma_Y + (A_Y \sigma_Y + \sigma_Y A_Y^T + C_Y \sigma_X C_Y^T) dt, \quad (\text{B8})$$

$$\begin{aligned} \xi_{X'Y'} &= (\sigma_X C_Y^T + C_X \sigma_Y) \sqrt{dt} \\ &:= B[\sigma_X] \sqrt{dt}. \end{aligned} \quad (\text{B9})$$

where B was defined in Eq. (24) and D is the diffusion matrix, defined in Eq. (22).

On the other hand, the conditional state of the system, given a certain measurement outcome, is still Gaussian, with first and second moments given by [59]

$$r_{X'|z} = r_{X'} + \xi_{X'Y'} (\sigma_{Y'} + \sigma_m)^{-1} (z - r_{Y'}), \quad (\text{B10})$$

$$\sigma_{X'|z} = \sigma_{X'} - \xi_{X'Y'} (\sigma_{Y'} + \sigma_m)^{-1} \xi_{X'Y'}^T. \quad (\text{B11})$$

Conditioning updates the average by a term proportional to the correlations $\xi_{X'Y'}$, as well as the outcomes z . Since z is random, $r_{X'|z}$ will be stochastic. The covariance matrix $\sigma_{X'|z}$, on the other hand, is reduced by the presence of the 2nd term

in (B11), called the Schur complement. Note that this term is, by construction, positive semi-definite, so that indeed conditioning always reduces the uncertainty about the system, as expected.

Eqs. (B10) and (B11) are general, in that they do not require the collision time to be infinitesimal. On the other hand, expanding in powers of dt and using Eqs. (B4)-(B9), we find

$$\begin{aligned} r_{X'|z} &= r_{X'} + B[\sigma_X](\sigma_Y + \sigma_m)^{-1/2}dw \\ &= r_X + Ar_X dt + B[\sigma_X](\sigma_Y + \sigma_m)^{-1/2}dw, \quad (\text{B12}) \\ \sigma_{X'|z} &= \sigma_{X'} - \chi[\sigma_X]dt \\ &= \sigma_X + (A\sigma_X + \sigma_X A^T + D - \chi[\sigma_X])dt, \quad (\text{B13}) \end{aligned}$$

where $dw = (\sigma_Y + \sigma_m)^{-1/2}(z - r_{Y'})$ can be shown to behave as a Wiener white noise term (that is, $\langle dw \rangle = 0$ and $\langle dw dw^T \rangle = dt I_Y$).

The first line in (B13) can be viewed as a manifestation of the so-called *law of total variance* [70]. In classical probability theory, the variance of a random variable X' can be written as

$$\text{var}(X') = \mathbb{E}_z \left(\text{var}(X'|z) \right) + \text{var}_z \left(\mathbb{E}(X'|z) \right). \quad (\text{B14})$$

The first term, called the *within-group* variation, measures the fluctuations $\text{var}(X'|z)$ within a given outcome z (i.e., within a given “group”), and then averaged it over all outcomes. Conversely, the second term, called *between-groups*, quantifies how much the conditional average $\mathbb{E}(X'|z)$ fluctuates between different outcomes z .

The law naturally extends for covariance matrices. And since we only condition on classical random variables z , the logic remains true, even though the system is quantum. The within group term is thus $\mathbb{E}_z(\sigma_{X'|z})$. But since $\sigma_{X'|z}$ doesn't depend on z , this simplifies to $\mathbb{E}_z(\sigma_{X'|z}) = \sigma_{X'|z}$. Similarly, the between groups is $\text{Cov}_z(r_{X'|z})$, where Cov stands for the

covariance matrix of the random vector $r_{X'|z}$. Hence, by comparison, moving $\chi[\sigma_X]dt$ to the left of Eq. (B13), we see that the between group contribution is precisely

$$\text{var}_z(r_{X'|z}) = \chi[\sigma_X]dt. \quad (\text{B15})$$

This provides another neat interpretation to the innovation matrix: It describes how $r_{X'|z}$ fluctuates between different outcomes z .

As a technical note, we mention that one could also, in principle, write down equations for the conditional state of the ancilla, given the measurement outcomes. That is, $r_{Y'|z}$ and $\sigma_{Y'|z}$. This, however, is not so easy, for it requires knowledge of the exact generalized measurement operators M_z . The noise covariance matrix σ_m , we are using here, only specifies the resulting POVM. And there is an infinite number of non-trivial choices of generalized measurements which yield the same POVM. Luckily, all quantities, both informational and thermodynamic, can be expressed without knowledge of $r_{Y'|z}$ and $\sigma_{Y'|z}$, as will be shown below.

Having established the evolution rules for a single collision, it is now straightforward to compound them and construction the continuous time dynamics. The unconditional dynamics, for instance, was given by the update rules (B4) and (B7) which, when adapted to multiple collisions, become:

$$r_{X_t} = r_{X_{t-1}} + Ar_{X_t} dt. \quad (\text{B16})$$

$$\sigma_{X_t} = \sigma_{X_{t-1}} + (A\sigma_{X_{t-1}} + \sigma_{X_{t-1}} A^T + D)dt, \quad (\text{B17})$$

Dividing by dt on both sides and taking the limit $dt \rightarrow 0$ then yields precisely Eqs. (23).

For the conditional dynamics, some care must be taken. Eqs. (B12) and (B13) refer to a single collision. Hence, the quantities r_X and σ_X that appear on the right-hand side, are actually the state of the system before that collision. In the case of a conditional dynamics, this would then be $r_{X_{t-1}|\zeta_{t-1}}$ and $\sigma_{X_{t-1}|\zeta_{t-1}}$. The left-hand side will then be associated with $X_t|\zeta_t$. Thus, the conditional evolution will be described by

$$r_{X_t|\zeta_t} = r_{X_{t-1}|\zeta_{t-1}} + Ar_{X_{t-1}|\zeta_{t-1}} dt + B[\sigma_{X_{t-1}|\zeta_{t-1}}](\sigma_Y + \sigma_m)^{-1/2}dw_t, \quad (\text{B18})$$

$$\sigma_{X_t|\zeta_t} = \sigma_{X_{t-1}|\zeta_{t-1}} + (A\sigma_{X_{t-1}|\zeta_{t-1}} + \sigma_{X_{t-1}|\zeta_{t-1}} A^T + D - \chi[\sigma_{X_{t-1}|\zeta_{t-1}}])dt. \quad (\text{B19})$$

Eq. (B18) leads to the Langevin equation (??), while Eq. (B19), when taking the limit $dt \rightarrow 0$, leads to the Riccati equation (??).

Appendix C: Examples of system-ancilla interactions in the continuous-variable case

In this appendix, we provide examples of some typical system-ancilla interactions in the continuous-variable sce-

nario. We also discuss the basic structure of the resulting matrices C_X , C_Y in Eq. (21), as well as the matrices A and D in Eqs. (22), which enter in many of the equations in Sec. III A.

1. Quantum-optical master equation

Suppose the system and ancilla are each comprised of a single mode of radiation, described by annihilation operators a_X

and a_Y and interacting with a beam-splitter Hamiltonian

$$\mathcal{H} = \omega(a_X^\dagger a_X + a_Y^\dagger a_Y) + i\sqrt{2\gamma}(a_X^\dagger a_Y - a_Y^\dagger a_X). \quad (\text{C1})$$

We introduce quadratures $q = (a_X + a_X^\dagger)/\sqrt{2}$ and $p = i(a_X^\dagger - a_X)/\sqrt{2}$ (and similarly for Q and P for the ancilla). The Hamiltonian then becomes of the form (20), with $H_X = H_Y = \omega I_2$ and

$$C = \begin{pmatrix} 0 & -\sqrt{2\gamma} \\ \sqrt{2\gamma} & 0 \end{pmatrix} \quad (\text{C2})$$

As a consequence, the matrices C_X and C_Y in Eq. (21) become

$$C_X = -C_Y = \sqrt{2\gamma}I_2, \quad (\text{C3})$$

so that A in (22) becomes

$$A = \omega\Omega_X - \gamma I_2 = \begin{pmatrix} -\gamma & \omega \\ -\omega & -\gamma \end{pmatrix}. \quad (\text{C4})$$

The interaction with the ancilla therefore introduces a damping term of intensity γ .

We also assume that the ancilla is initially thermal,

$$\sigma_Y = (\bar{n} + 1/2)I_2, \quad (\text{C5})$$

where \bar{n} is the thermal occupation. The diffusion matrix D then becomes

$$D = C_X \sigma_Y C_X^\dagger = \gamma(2\bar{n} + 1)I_2. \quad (\text{C6})$$

The resulting unconditional dynamics, compounding the effects of multiple collisions, therefore corresponds to the usual quantum optical master equation

$$\frac{d\rho_X}{dt} = -i[H_X, \rho_X] + \gamma(\bar{n} + 1)\mathcal{D}[a_X] + \gamma\bar{n}\mathcal{D}[a_X^\dagger],$$

where $\mathcal{D}[L] = L\rho_X L^\dagger - \frac{1}{2}\{L^\dagger L, \rho_X\}$.

2. Position-position coupling

Next we consider the case in which the system and ancilla are still given by a single mode each, but now coupled through an interaction of the form

$$\mathcal{H}_{\text{int}} = -\sqrt{g}qQ. \quad (\text{C7})$$

The interaction matrix C in Eq. (20) becomes

$$C = -\sqrt{g} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (\text{C8})$$

so that

$$C_X = C_Y = \sqrt{g} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (\text{C9})$$

Quite interestingly we see that in this case $C_X C_Y = C_Y C_X = 0$. Hence, the drift terms in Eqs. (22) and (B3) vanish completely. The diffusion matrix, on the other hand, becomes

$$D = g(\bar{n} + 1/2) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

A position-position coupling therefore introduces diffusion only in the momentum quadrature (and no damping in either).

3. Ancillae with multiple components

As a final example, let us consider the case where each interaction actually involves an ancilla with two components, $Y = (Q_1, P_1, Q_2, P_2)$. This helps to gain intuition about the sizes of the matrices. It is also important when only some of the ancillae are actually measured, which is an experimentally meaningful hypothesis: normally, the system will interact with many ancillae at once, but the experimenter may have access to only some of them.

The interaction matrix C in Eq. (20) now becomes rectangular:

$$C = (C_1 \ C_2), \quad (\text{C10})$$

with C_1 and C_2 being 2×2 matrices. The matrices C_X and C_Y , in turn, become,

$$C_X = (C_{X_1} \ C_{X_2}), \quad C_Y = \begin{pmatrix} C_{Y_1} \\ C_{Y_2} \end{pmatrix}, \quad (\text{C11})$$

where C_{X_i} and C_{Y_i} are all 2×2 . For instance, if (Q_1, P_1) interacts with the system according to a beam-splitter interaction (C1), then C_1 will be given exactly by Eq. (C2) and C_{X_1} , C_{Y_1} will be given by Eq. (C3).

Appendix D: Calculation of information and thermodynamic quantities in the continuous variable scenario

This appendix provides details on the calculation of information-theoretic and thermodynamic quantities in the case of continuous variable models. To do so we will use some of the results of Appendix B. The information rate (??) is given by

$$\Delta I_t = \frac{1}{2} \ln \frac{|\sigma_{X_t}|}{|\sigma_{X_{t-1}}|} - \frac{1}{2} \ln \frac{|\sigma_{X_t|\zeta_t}|}{|\sigma_{X_{t-1}|\zeta_{t-1}}|}. \quad (\text{D1})$$

But to compute G_t and L_t in Eq. (10) we need $I(X_t : \zeta_{t-1})$. This is obtained from the map (??) which, in the language of covariance matrices, consists in applying the unconditional evolution (B13) to $\sigma_{X_{t-1}|\zeta_{t-1}}$; viz.,

$$\sigma_{X_t|\zeta_{t-1}} = \sigma_{X_{t-1}|\zeta_{t-1}} + (A\sigma_{X_{t-1}|\zeta_{t-1}} + \sigma_{X_{t-1}|\zeta_{t-1}}A^\dagger + D)dt. \quad (\text{D2})$$

We then immediately get, using the definitions (8) and (9),

$$G_t = \frac{1}{2} \ln \frac{|\sigma_{X_t|\zeta_t}|}{|\sigma_{X_t|\zeta_{t-1}}|}, \quad (\text{D3})$$

$$L_t = \frac{1}{2} \ln \frac{|\sigma_{X_{t-1}}|}{|\sigma_{X_t}|} - \frac{1}{2} \ln \frac{|\sigma_{X_{t-1}|\zeta_{t-1}}|}{|\sigma_{X_t|\zeta_{t-1}}|}. \quad (\text{D4})$$

As a sanity check, subtracting $G_t - L_t$ clearly leads to (D1).

Eqs. (D1), (D3) and (D4) do not assume infinitesimal collisions. To obtain a continuous time description, we expand the determinants to leading order in dt . To do this, the following

result turns out to be quite useful: Consider the Wigner entropy (30) and assume that $\sigma = \sigma_0 + \sigma_1$, where σ_1 is small. A series expansion of $|\sigma|$ in powers of σ_1 then yields,

$$\frac{1}{2} \ln |\sigma| = \frac{1}{2} \ln |\sigma_0| + \frac{1}{2} \text{tr}(\sigma_0^{-1} \sigma_1) - \frac{1}{4} \text{tr}(\sigma_0^{-1} \sigma_1 \sigma_0^{-1} \sigma_1) + \dots \quad (\text{D5})$$

This is useful expression because all results just presented are of this form.

We begin by applying it to Eq. (D1). First, from (B17) we get

$$\frac{1}{2} \ln \frac{|\sigma_{X_t}|}{|\sigma_{X_{t-1}}|} = \frac{1}{2} \text{tr}(2A + \sigma_{X_{t-1}}^{-1} D) dt,$$

where we used the fact that $\text{tr}(A) = \text{tr}(A^T)$. Similarly, Eq. (B19) yields

$$\frac{1}{2} \ln \frac{|\sigma_{X_t|Z_t}|}{|\sigma_{X_{t-1}|Z_{t-1}}|} = \frac{1}{2} \text{tr} \left\{ 2A + \sigma_{X_{t-1}|Z_{t-1}}^{-1} D - \sigma_{X_{t-1}|Z_{t-1}}^{-1} \chi[\sigma_{X_{t-1}|Z_{t-1}}] \right\} dt.$$

In the limit $dt \rightarrow 0$, Eq. (D1) therefore reduces to the result in Eq. (32). Similarly, repeating the procedure for G_t and L_t in Eqs. (D3) and (D4) and identifying $\dot{G} = G_t/dt$ and $\dot{L} = L_t/dt$, leads to Eqs. (33) and (34).

To compute the thermodynamic quantities, we first note that the relative Wigner entropy between two Gaussian states ρ_1 and ρ_2 , with covariance matrices σ_1 and σ_2 , and first moments r_1 and r_2 , can be written as [65]

$$D(\rho_1 \|\rho_2) = \frac{1}{2} \text{tr}[\sigma_2^{-1}(\sigma_1 - \sigma_2)] + S(\sigma_2) - S(\sigma_1) \quad (\text{D6}) \\ + \frac{1}{2}(r_1 - r_2)^T \sigma_2^{-1} (r_1 - r_2).$$

Thus, the entropy flux (??) in a single collision can be written

as

$$\Delta\Phi_t = \frac{1}{2} \text{tr}[\sigma_{Y_t}^{-1}(\sigma_{Y_t'} - \sigma_{Y_t})] + \frac{1}{2} r_{Y_t}^T \sigma_{Y_t}^{-1} r_{Y_t'}. \quad (\text{D7})$$

Plugging in Eqs. (B5) and (B8) then leads to Eq. (37), where $\dot{\Phi} = \Delta\Phi_t/dt$. We also need to use the fact that

$$\text{tr}(A_Y) = \text{tr}(A) = \frac{1}{2} \text{tr}(C_X C_Y), \quad (\text{D8})$$

which follows from Eqs. (??) and (B3), together with the fact that $\Omega_X H_X$ and $\Omega_Y H_Y$ are traceless. Finally, to obtain $\dot{\Sigma}^u$, we use again the expansion (D5) to

$$S(X_t) = S(X_{t-1}) + \frac{1}{2} \text{tr}\{2A + \sigma_{X_{t-1}}^{-1} D\} dt. \quad (\text{D9})$$

Combining this with (D7) and taking the limit $dt \rightarrow 0$ then leads to Eq. (38).

We finish with a technical note. In deriving these expressions we have tacitly assumed that Eq. (??) is satisfied. In general, however, there is no guarantee that the state of the ancilla after the (non-selective) measurement, $\tilde{\rho}_{Y'} = \sum_z M_z \rho_{Y'} M_z$, will still be Gaussian. This is due to the fact that, despite the POVM of a (noiseless) general-dyne measurement corresponds to the PVM over some (pure) Gaussian state, there are infinitely many (unitarily equivalent) quantum operations corresponding to the same POVM and some of these operations can give rise to a non-Gaussian $\tilde{\rho}_{Y'}$. It is however easy to see that, any time the state of the ancilla $\tilde{\rho}_{Y'}$ is Gaussian, then the entropy flux is well-defined in terms of the Wigner relative entropy and Eq. (??) is automatically satisfied. Physically speaking, this is always the case when we assume the operations acting on the ancilla to be the projections over Gaussian states. Moreover, since the state of the ancilla after the measurement is rarely of interest, and in accordance with the classical intuition spelled out in ??, the possible mismatch between the conditional and unconditional fluxes can be safely neglected.

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