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A new value for Cooperative Games based on coalition size

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Abstract

We propose and characterize a new value for TU cooperative games based on egalitarian distribution of worths in smaller coalitions and players marginal productivity in larger coalitions. This value belongs to the class of Procedural values due to (Malawski, 2013). Our value is identical with the Shapley value on one extreme and the Equal Division rule on the other extreme. We show that our value is identical with the solidarity value due to (Bèal et al., 2017) of the dual game. However, by duality, our characterization intuitively improves over the axiomatization of this solidarity value. We also provide a mechanism that implements our value in sub-game perfect Nash equilibrium. Finally, a generalized version of this value is proposed followed by its characterizations.

Keywords: Shapley value; Equal Division rule; Solidarity; Egalitarian Shapley value.
MSC(2010): 91A12; JEL: C71, D60.

1 Introduction

Cooperative games with transferable utilities or simply TU games describe situations where a finite set of players make binding agreements to generate worths and share it among themselves. Their applications in economic allocation problems are wide and varied (see for instance [18, 2])[1]. The underlying assumption is that the players form the grand coalition under such binding agreements. A solution comprises distributions of the worth of the grand coalition among the players. A singleton solution is called a value. Many of the values found in the literature revolve around the notions of egalitarianism and marginalism. Among all the values found in the literature, the Shapley value [28] is perhaps the most popular one that builds on the notion of marginalism [32]. It is the expectation of the increase in transferable utilities of a player when she joins a coalition [31], which we call her marginal contribution. Unlike the Shapley value, the Equal Division(ED) rule allocates the worth of the grand coalition equally among all the players and can be considered to be the most egalitarian solution for TU games. Values that combine both these attributes are said to be based on solidarity principles outlined in [11,12] and therefore, we broadly classify all such values as solidarity values. The authors in [13, 21] axiomatize the class of Egalitarian Shapley values which are obtained as convex combinations of the ED and the Shapley value. The convexity parameter α in the Egalitarian Shapley values determines how much solidarity is allowed to the players. In [1] the equal division is generalized to obtain a weighted division and the corresponding axiomatization is obtained. For a comprehensive study of all these values, we refer to [3, 5, 6, 10, 11, 15, 24, 22].

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[1] More recently the network allocation rules (see, e.g., [7, 8]) have found applications in Bioinformatics (see, e.g., [4] and the references therein).

Except [3] however, none of the solidarity values hitherto found in the literature treats the coalition size as a parameter. In this paper, we propose a new solidarity value for TU games that allows for egalitarianism in smaller coalitions and marginalism in larger coalitions. The proposed solidarity value combines egalitarianism characteristic of the ED and marginalism characteristic of the Shapley value based on the threshold size k of coalitions at which the norms change from egalitarianism to marginalism, and therefore it is termed as the k-Equal Division Shapley value or the k-EDS value in short. The k-EDS value belongs to a special class of the Procedural values due to [23]. We provide a characterization of this value on the basis of coalitional sizes. We also propose a bidding mechanism to implement our value as a sub-game perfect Nash equilibrium of the associated non-cooperative game that arises naturally from the bargaining framework generated by the mechanism [3, 9, 31]. Finally, we provide a generalized form of values that includes the k-EDS value and several other existing values found in the literature.

In society, egalitarianism is often observed in small coalitions while marginalism is typically observed in large coalitions. An excellent argument for this can be found in [27], where the author comments that since the beginning of the civilization, humans have been mostly living as a species in relatively egalitarian, small-scale societies. The ecological and demographic conditions common to small-scale societies favored the suppression of steep, dominance-based hierarchies and incentivized relatively shallow, prestige-based hierarchies. Shifts in ecological and demographic conditions, particularly with the spread of agriculture i.e., through adding more people into the groups, weakened constraints on coercion. Following this line of arguments, it is not hard to see that in a coalition of sufficiently small size for example, within a community or a tribe or clan, players are more friendly, homogenous, and socially involved and hence more often they follow the egalitarian sharing of the resources among themselves. However, when more people enter the coalition making it sufficiently large and heterogeneous, productive players prefer not to share their earnings equally with non-productive or less productive players.

Motivated by the above discussion, in this paper we first propose our value that embodies both marginalism and egalitarianism depending on the size of the coalitions. Then, we characterize this value using standard and intuitive axioms. We call our value: the k-EDS value that guarantees egalitarian shares to the non-productive players within small groups like families, communities, and tribes. The players would unanimously decide the fixed size k of these small groups. On the other hand, it also assures that more productive players are not deprived of their marginal productivities in sufficiently larger coalitions.

We give a characterization of the k-EDS value along the line of [10, 30]. This characterization provides alternative interpretations of the k-EDS value relevant to the social and economic issues. Note that the converse of our model is studied by [3] where marginal contributions are assigned to small coalitions and equal divisions to large coalitions. The authors [3] propose the Sol^p value based on the threshold size p of the coalition at which the role switches from marginalism to egalitarianism. We have shown that the k-EDS value of a TU game is the Sol^p value of its dual game. However, the characterization of the k-EDS value provides clearer insights than the Sol^p value because of the intuitive nature of the axioms that characterize it. We have a separate section i.e., Section 5 where we compare the two models.

Next, we allow for reconciliation of marginalism and egalitarianism for both the groups of players separated by size k by awarding them the α-Egalitarian Shapley value at two levels i.e., α_1 to all the coalitions of size less than k and α_2 to all the coalitions of size greater than or equal to k. This will be a generalized version of the k-EDS value as the ED and the Shapley value are two special cases of the Egalitarian Shapley value.

The rest of the paper proceeds as follows. In Section 2 we present the preliminary definitions and results pertaining to the development of the paper. Section 3 describes a procedure to compute the k-EDS value followed by its characterization in Section 4. Section 5 provides a comparison of the k-EDS value with some existing values. In Section 6, we present a bidding mechanism to implement the k-EDS value. Section 7 generalizes the class of k-EDS values and finally, Section 8 concludes.

2 Preliminaries

Let $\mathcal{N}$ be the universal set of players and $N = \{1, 2, ..., n\} \subseteq \mathcal{N}$ be a finite set of n players. Also let 2^N denote the power set of N. The subsets of N are called coalitions. Denote the size of a coalition S by the corresponding lower case letter s. To simplify notation, we write $S \cup i$ for $S \cup \{i\}$ and $S \setminus i$ for $S \setminus \{i\}$ for each $S \subseteq N$ and $i \in N$. A cooperative game or simply a TU game is a pair (N, v) where the function $v : 2^N \to \mathbb{R}$ is such that $v(\emptyset) = 0$. For each $S \subseteq N$, $v(S)$ denotes the worth of the coalition S. Denote by v_0 the *null game*, defined as $v_0(S) = 0$ for all $S \subseteq N$. A TU game is *zero-monotonic* if, for each $i \in N$ and each coalition $S \ni i$, it holds that $v(S) - v(S \setminus i) \geq v(i)$. The class of all TU games over the universal player set $\mathcal{N}$ is denoted by G and the subclass of TU games over the player set N is denoted by $G(N)$. Thus, the members of G are taken as (N, v), where $N \subset \mathcal{N}$ and the members of $G(N)$ are taken simply by the function v as N is fixed here. Recall that the standard assumption of TU games is that the grand coalition is eventually formed. A solution of an n-player TU-game is an n-dimensional payoff vector $x \in \mathbb{R}^n$ giving a payoff $x_i \in \mathbb{R}$ to every player $i \in N$. A value on $G(N)$ is a function Φ that assigns a payoff vector $\Phi(v) \in \mathbb{R}^n$ to each $v \in G(N)$ for a fixed player set N. The class $G(N)$ of all TU games with player set N forms a vector space of dimension $2^n - 1$ under the standard addition and scalar multiplication of set functions. For every coalition $S \subseteq N$ with $S \neq \emptyset$, the games $e_S : 2^N \to \mathbb{R}$ and $u_S : 2^N \to \mathbb{R}$ given by,

$$e_S(T) = \begin{cases} 1, & \text{if } T = S \\ 0, & \text{otherwise,} \end{cases} \tag{2.1}$$

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq T \\ 0, & \text{otherwise,} \end{cases} \tag{2.2}$$

are standard bases for the class $G(N)$ of TU games with player set N called the identity games and the unanimity games respectively. For every game $v \in G(N)$, we can write $v = \sum_{S \neq \emptyset} v(S) e_S$ and $v = \sum_{S \neq \emptyset} \Delta_S(v) u_S$ where $\Delta_S(v) = \sum_{T \subseteq S} (-1)^{s-t} v(T)$. The marginal contribution of player i to coalition $S \subseteq N \setminus i$ is formally written as,

$$m_i^v(S) = v(S \cup i) - v(S). \tag{2.3}$$

Suppose that the grand coalition N is formed in such a way that the players enter the coalition one by one. Such entry can be determined by a permutation $\pi : N \to N$ of the players. We denote the collection of all permutations by $\Pi(N)$. For every $\pi \in \Pi(N)$, we denote by $P(\pi, i) = \{j \in N : \pi(j) < \pi(i)\}$ the set of players that enter before player i in the permutation π. The Shapley value [28] denoted by Φ^{Sh} assigns to every player her expected marginal contribution (to the coalition of players that enters before her), given that every permutation of entrance π has equal probability of occurrence namely, $\frac{1}{n!}$. Therefore, the Shapley value is given by

$$\Phi_i^{Sh}(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m_i^v(P(\pi, i)), \tag{2.4}$$

which after simplifications becomes

$$\Phi_i^{Sh}(v) = \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!} m_i^v(S). \tag{2.5}$$

The Equal Division (ED) rule is a solution $\Phi^{ED} : G(N) \to \mathbb{R}^n$ that distributes the worth $v(N)$ of the grand coalition equally among all players in any game, i.e.,

$$\Phi_i^{ED}(v) = \frac{v(N)}{n}. \tag{2.6}$$

It follows from (2.5) and (2.6) that both the Shapley value and the Equal Division rule can be expressed in a unified manner as follows:

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!} C_i^v(S), \tag{2.7}$$

where $C_i^v : 2^{N\setminus i} \mapsto \mathbb{R}$ is such that for $S \subseteq N \setminus i$, $C_i^v(S) = m_i^v(S)$ when $\Phi = \Phi^{Sh}$ and $C_i^v(S) = e_N(S \cup i)v(S \cup i)$ when $\Phi = \Phi^{ED}$. Call $C_i^v(S)$ the coalitional contribution of player i in S with respect to $v \in G(N)$. Thus, under this new notation, both the Shapley value and the ED assign to each player her expected coalitional contribution $C_i^v(S)$ where $C_i^v(S) = m_i^v(S)$ for each $S \subseteq N \setminus i$ in case of the Shapley value and $C_i^v(S) = 0$ for $S \subset N \setminus i$, and $C_i^v(N \setminus i) = v(N)$ in case of the Equal Division rule.
In [23], the class of Procedural values is introduced. A Procedural value is determined by an underlying procedure of sharing the marginal contributions with coalitions formed by players joining in random orders. The class of Procedural values includes the Shapley value, the Equal Division rule and many other solidarity values. In the following, we give the formal definition of a Procedural value.

Definition 1. [23] A procedure γ on $G(N)$ is a family of non-negative coefficients $((\gamma_{k,j})_{j=1}^k)_{k=1}^n$ such that $(\forall k) \sum_{j=1}^k \gamma_{k,j} = 1$. The coefficient $\gamma_{k,j}$ describes the share of the player who is at place j in the order, i.e., if π is an order (a permutation on N) then γ_{kj} describes the share of player $\pi^{-1}(j)$ from the marginal contribution of player $\pi^{-1}(k)$, where $\pi^{-1}(l)$ represents the position of player $l \in N$ in the order π. The Procedural value ψ^γ determined by the procedure γ on $G(N)$ is defined by the formula:

$$\psi_i^\gamma(v) = \sum_{\pi \in \Pi(N)} \sum_{j \in N(\pi,i)} \frac{\gamma_{\pi(j)\pi(i)} m_j^v(P(\pi, j))}{n!} \tag{2.8}$$

where $N(\pi, i) = \pi^{-1}(\{\pi(i), \pi(i)+1, \cdots, n\})$ is the set of the successors of i including i in π.

The following two procedures that respectively lead to the Shapley value and the Equal Division rule are important for us.
Procedure 1 $\forall k(\gamma_{k,k} = 1 \text{ and } \forall j < k,\ \gamma_{k,j} = 0)$. This procedure leads to the Shapley value Φ^{Sh}.
Procedure 2 $(\gamma_{1,1} = 1; \text{ and } \forall k > 1,\ \gamma_{k,k-1} = 1)$. This procedure leads to the Equal Division rule Φ^{ED}.

In [21] the α-Egalitarian Shapley value is introduced which is a convex combination of the ED and the Shapley value. Thus formally we have the following. For $\alpha \in [0, 1]$, the α-Egalitarian Shapley value due to [21] is given by

$$\Phi_i^{\alpha-\text{ES}}(v) = \alpha\Phi_i^{ED}(v) + (1-\alpha)\Phi_i^{Sh}(v). \tag{2.9}$$

Various axiomatizations of the Shapley value, the Equal Division rule and the α-Egalitarian Shapley value can be found in the literature (see [17, 21, 28, 29, 30, 32]). In the following, we list some of the important axioms that characterize these values and also are relevant for the current paper. But first we define the following:

Definition 2. A player $i \in N$ is a null player in $v \in G(N)$ if $m_i^v(S) = 0$ for every coalition $S \subseteq N \setminus i$.

Definition 3. A player $i \in N$ is a nullifying player in $v \in G(N)$ if $v(S) = 0$ for every coalition S where $i \in S$.

Definition 4. Two players $i, j \in N$ are called symmetric with respect to the game v if for all $S \subseteq N \backslash \{i, j\}$,

$$v(S \cup i) = v(S \cup j).$$

We list the axioms for a value $\Phi : G(N) \to \mathbb{R}^n$ as follows:

Axiom 1. Efficiency (Eff): For all $v \in G(N) : \sum_{i \in N} \Phi_i(v) = v(N)$.

Axiom 2. Null Player Property (NP): For every game $v \in G(N)$ and every null player $i \in N$ in v, we have $\Phi_i(v) = 0$.

Axiom 3. Nullifying Player Property (NPP): For every game $v \in G(N)$ and every nullifying player $i \in N$ in v, $\Phi_i(v) = 0$.

Axiom 4. Symmetry (Sym): For every pair of symmetric players $i, j \in N$ with respect to the game $v \in G(N)$, we have $\Phi_i(v) = \Phi_j(v)$.

Axiom 5. Linearity (Lin): For all games $u, w \in G(N)$, every pair of $\gamma, \eta \in \mathbb{R}$, and every player $i \in N$:

$$\Phi_i(\gamma u + \eta w) = \gamma \Phi_i(u) + \eta \Phi_i(w); \tag{2.10}$$

Φ is additive (ADD) if in particular (2.10) holds for $\gamma = \eta = 1$.

Axiom 6. Strong monotonicity (SMon): $\Phi_i(v) \geq \Phi_i(w)$ for every pair of games $v, w \in G(N)$ and player $i \in N$ such that $m_i^v(S) \geq m_i^w(S)$ for all $S \subseteq N \setminus i$.

Axiom 7. Coalitional strategic equivalence (CSE): For every pair of games $v, w \in G(N)$, a value Φ satisfies $\Phi_i(v + w) = \Phi_i(v)$ whenever i is a null player in w.

Axiom 8. Marginality (M): For all $v, w \in G(N)$, $i \in N$, $m_i^v(S) = m_i^w(S)$ for all $S \subseteq N \setminus i$ implies $\Phi_i(N, v) = \Phi_i(N, w)$.

Axiom 9. Fairness[2] (F): For any two symmetric players $i, j \in N$ in $w \in G(N)$, it holds that

$$\Phi_i(v + w) - \Phi_i(v) = \Phi_j(v + w) - \Phi_j(v), \qquad \forall v \in G(N).$$

Axiom 10. α-Egalitarian (α-E): The value Φ is α-Egalitarian if for each $v \in G(N)$, $\Phi_i(v) = \alpha \frac{\sum_{j \in N} \Phi_j(v)}{n}$ whenever $i \in N$ is a null player in v.

The most standard characterization of the Shapley value requires Eff, Sym, Lin, and NP. The ED, on the other hand, has been characterized using Eff, Sym, Lin, and NPP (see [30]). An alternative characterization of the Shapley value is due to [32] using Eff, Sym, and (SMon or M[3]). Chun [17] characterizes the Shapley value using Eff, Sym, and CSE. van den Brink [29] characterizes the Shapley value by the axioms of Eff, NP, and F. In [21] it is shown that the Egalitarian Shapley value can be characterized by Eff, Sym, ADD and α-E. Some alternative characterizations using other axioms can be found in [29, 30, 32].

A value that satisfies Eff, Sym and Lin is called an ESL value [20]. We mention the following proposition on the class of ESL values from [26] for later reference.

Proposition 1. *(Proposition 2 in [26], p. 184)* A value Φ on $G(N)$ is an ESL-value if and only if there exists a unique collection of real constants $B^\Phi = (b_s^\Phi : s \in \{0, 1, 2, ..., n\})$ with $b_n^\Phi = 1$ and $b_0^\Phi = 0$ such that for every game $v \in G(N)$,

$$\Phi_i(v) = \sum_{S \subseteq N \backslash i} \frac{s!(n - s - 1)!}{n!} \left\{ b_{s+1}^\Phi v(S \cup i) - b_s^\Phi v(S) \right\}. \tag{2.11}$$

[2] In [10] this axiom is termed as *van den Brink fairness* since it was introduced by van den Brink in [29].
[3] Young [32] calls this the independence property.

That is

$$\Phi_i(v) = \Phi_i^{Sh}(B^{\Phi}v) \tag{2.12}$$

where $(B^{\Phi}v)(S) = b_t^{\Phi}v(S)$ for each coalition S of size s.

3 The k-EDS value

We now introduce our value for TU games, which we call the k-EDS value. We follow an approach similar to Shapley's [28] approach where the players are allowed to enter a coalition following a particular permutation assuming that all possible permutations of entrance have equal probabilities. Recall that our value is based on the assumption that given a coalition S of sufficiently small size, each player i agrees to the egalitarian distribution of its worth $v(S)$, namely $\frac{v(S)}{s}$: let us call it the egalitarian coalitional contribution. However, when the size of the coalitions is sufficiently large, the coalitional contributions become marginal contributions $m_i^S(v)$ and are no longer egalitarian.

Let k be the maximum size of the coalitions in which each player enjoys egalitarian coalitional contributions. Then, for each $S \subseteq N$, we define the quantity $C_i^{(k,v)}(S)$ by

$$C_i^{(k,v)}(S) = \begin{cases} \dfrac{v(S)}{k}, & \text{if } s \leq k \\ v(S \cup i) - v(S), & \text{if } s > k. \end{cases} \tag{3.1}$$

Let us call $C_i^{(k,v)}(S)$ the k-coalitional contributions of player i. Let us now assume that the players enter the coalition one by one. Define $P^k(\pi) = \{j \in N | \pi(j) \leq k\}$. Then $P^k(\pi)$ represents the set of first k players who enter the game under the permutation π. If $\pi(i) \leq k$ then $P(\pi, i) \cup i \subseteq P^k(\pi)$. Thus, following our assumption the k-coalitional contribution $C_i^v(P(\pi, i))$ of player i is $\dfrac{v(P^k(\pi))}{k}$ when i joins $P(\pi, i)$ and $\pi(i) \leq k$. On the other hand, when player i joins coalition $P(\pi, i)$ with $\pi(i) > k$, then $C_i^v(P(\pi, i)) = v(P(\pi, i) \cup i) - v(P(\pi, i))$ for the permutation π. Following Eq.(3.1), the k-coalitional contribution $C_i^{(k,v)}(P(\pi, i))$ of player i in a game v with respect to the parameter k in forming the grand coalition N following the permutation π is given by

$$C_i^{(k,v)}(P(\pi, i)) = \begin{cases} \dfrac{v\Big(P^k(\pi)\Big)}{k}, & \text{if } \pi(i) \leq k \\ v(P(\pi, i) \cup i) - v(P(\pi, i)), & \text{if } \pi(i) > k. \end{cases} \tag{3.2}$$

Definition 5. The k-EDS value denoted by $\Phi^{k-\mathrm{EDS}} : G(N) \to \mathbb{R}^n$ is the value that assigns to every player $i \in N$, her expected coalitional contribution given by

$$\Phi_i^{k-\mathrm{EDS}}(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} C_i^{(k,v)}(P(\pi, i)) \tag{3.3}$$

where $C_i^{(k,v)}(P(\pi, i))$ is given by (3.2).

The expression for $\Phi^{k-\mathrm{EDS}}(v)$ in 3.3 can be simplified as follows:

$$\begin{aligned} \Phi_i^{k-\mathrm{EDS}}(v) &= \frac{1}{n!} \sum_{\pi \in \Pi(N)} C_i^{(k,v)}(P(\pi, i)) \\ &= \frac{1}{n!} \sum_{\pi \in \Pi(N):\pi(i) \leq k} \frac{v(P^k(\pi))}{k} + \frac{1}{n!} \sum_{\pi \in \Pi(N):\pi(i) > k} \Big\{ v(P(\pi, i) \cup i) - v(P(\pi, i)) \Big\} \\ &= \frac{1}{n!} \sum_{\pi \in \Pi(N):|P(\pi, i)| = k-1} v(P(\pi, i) \cup i) + \frac{1}{n!} \sum_{\pi \in \Pi(N):|P(\pi, i)| \geq k} \Big\{ v(P(\pi, i) \cup i) - v(P(\pi, i)) \Big\} \end{aligned} \tag{3.4}$$

On further simplification, 3.4 can be re-written as follows:

$$\Phi_i^{k-\text{EDS}}(v) = \sum_{\substack{S\subseteq N\,:\,i\notin S\\ s=k-1}} \frac{(n-k)!(k-1)!}{n!} v(S\cup i) + \sum_{\substack{S\subseteq N\,:\,i\notin S\\ s\geq k}} \frac{(n-s-1)!s!}{n!} \Big\{ v(S\cup i) - v(S) \Big\}. \qquad (3.5)$$

Observation 1.

(a) The k-EDS value coincides with the Shapley value only for $k = 1$ and with ED only for $k = n$.

(b) For each $v \in G(N)$, define $v_{\geq k} \in G(N)$ as follows.

$$v_{\geq k}(S) = \begin{cases} 0, & \text{if } s < k \\ v(S), & \text{if } s \geq k \end{cases} \qquad (3.6)$$

Then

$$\Phi_i^{k-\text{EDS}}(v) = \Phi_i^{Sh}(v_{\geq k}). \qquad (3.7)$$

(c) Generalizing (b) above, we have for each $v \in G(N)$, $\Phi^{k-\text{EDS}}(v) = \Phi^{Sh}(B^k v)$ where $B^k = (b_s : s \in \{0, 1, 2, ..., n\})$ such that $b_0 = 0$, $b_s = 0$ for $s < k$, $b_s = 1$ for $s \geq k$.

(d) For each $v \in G(N)$, $\Phi^{k-\text{EDS}}(v) = \psi^{\gamma}(v)$, the Procedural value with procedure γ where $\gamma_{1,1} = 1$, and

$$\begin{aligned} \gamma_{r,r-1} &= 1 \text{ whenever } 1 < r \leq k \\ \gamma_{r,r} &= 1 \text{ whenever } r > k. \end{aligned}$$

Following is an example that illustrates how the k-EDS value combines marginalism with egalitarianism for different values of k.

Example 1. Take $N = \{1, 2, 3\}$. Define $v : 2^{\{1,2,3\}} \mapsto \mathbb{R}$ by $v(\{1\}) = 0$, $v(\{1, 2\}) = v(\{2\}) = 5$, $v(\{1, 3\}) = v(\{3\}) = 10$, and $v(\{1, 2, 3\}) = v(\{2, 3\}) = 18$. Note that, here player 1 is a null player. The k-EDS values for different values of k are given in Table 1.

Observe from Table 1, that for $k = 2$, each player gets the egalitarian payoff from her coalitions of size

Values	k	Player 1	Player 2	Player 3	Total = $v(N)$
$\Phi^{ED}(v)$	3	6	6	6	18
$\Phi^{k}(v)$	2	2.5	6.5	9	18
$\Phi^{Sh}(v)$	1	0	7.33	10.67	18

Table 1: k-EDS values for different values of k in Example 1.

2, while from coalitions of size 3 (that is, the grand coalition in this case) she gets her Shapley payoff. Further, the null player 1 receives 0 payoff under the Shapley value ($k = 1$) and egalitarian payoff of 6 under the ED ($k = 3$). For $k = 2$, her payoff is a consolidation of marginalism and egalitarianism.

Generalizing the k-EDS value to include all possible values of $k \in \{1, 2, ..., n\}$ in the line of [3], we define the α-EDS value as follows. Assume that the integer k is drawn from the set $\{1, 2, 3, ..., n\}$ according to the probability distribution $\alpha = (\alpha_k : k \in \{1, 2, 3, ..., n\})$. Then the α-EDS value $\Phi^{\alpha-\text{EDS}}$ induced by the probability distribution α is defined for $v \in G(N)$ as the expected payoff given by

$$\Phi^{\alpha-\text{EDS}}(v) = \sum_{k=1}^{n} \alpha_k \Phi^{k-\text{EDS}}(v) \qquad (3.8)$$

Thus the α-EDS value computes the expected payoff of each player under the probability distribution α. We have the following representation theorem of the α-EDS value which will be later used for comparing it with the existing solidarity values.

Proposition 2. A value Φ is an α-EDS value on $G(N)$ if and only if it can be represented by an ESL value i.e., by Eq.(2.11) or equivalently by Eq.(2.12) with constants $B^{\Phi} = (b_s^{\Phi} : s \in \{0, 1, 2, 3, ..., n\})$ such that $0 = b_0^{\Phi} \leq b_1^{\Phi} \leq ... \leq b_{n-1}^{\Phi} \leq b_n^{\Phi} = 1$, with at least one inequality being strict.
Further, $\Phi = \Phi^{\alpha-\text{EDS}}$ where $\alpha = \{\alpha_s : s \in \{0, 1, 2, ..., n\}\}$ is obtained from the transformation $B^{\Phi} \mapsto \alpha$ such that $\alpha_0 = b_0^{\Phi} = 0$, $\alpha_s = b_s^{\Phi} - b_{s-1}^{\Phi}, \ \ \forall s \in \{1, 2, 3, ..., n\}$.

Proof. Assume that Φ be an α-EDS value. Therefore, there exists a probability distribution $\alpha = (\alpha_k : 1 \leq k \leq n)$ such that

$$\Phi_i(v) = \sum_{k=1}^{n} \alpha_k \Phi_i^{k-\text{EDS}}(v) \quad \text{for all} \ \ v \in G(N)$$

Following (c) in observation 1(c), we have

$$\Phi_i(v) = \sum_{k=1}^{n} \alpha_k \Phi_i^{Sh}(B^k v) \quad \text{for all} \ \ v \in G(N)$$

where $B^k = (b_s : s \in \{0, 1, 2, ..., n\})$ such that $b_0 = 0$, $b_s = 0$ for $s < k$, and $b_s = 1$ for $s \geq k$. by Lin of Φ^{Sh} we have,

$$\begin{aligned}\Phi_i(v) &= \Phi_i^{Sh}\left(\sum_{k=1}^{n} \alpha_k B^k v\right) \quad \text{for all} \ \ v \in G(N) \ , i \in N \\ &= \Phi_i^{Sh}\left(\left\{\sum_{k=1}^{n} \alpha_k B^k\right\} v\right)\end{aligned}$$

Define a vector B^{Φ} given by $B^{\Phi} = \sum_{k=1}^{n} \alpha_k B^k$. Since $B^k = (b_s : s \in \{0, 1, 2, ..., n\})$ such that $b_0 = 0$, $b_s = 0$ for $s < k$, and $b_s = 1$ for $s \geq k$, we have, $b_0^{\Phi} = 0$, $b_s^{\Phi} = \sum_{k=1}^{s} \alpha_k$ for $s \geq 1$. Since $\alpha_k \geq 0$ and $\sum_{k=1}^{n} \alpha_k = 1$, therefore, we have $0 = b_0^{\Phi} \leq b_1^{\Phi} \leq b_2^{\Phi} \leq ... \leq b_n^{\Phi} = 1$. It follows that, $\Phi(v) = \Phi^{Sh}(B^{\Phi} v)$ where $B^{\Phi} = \{b_s^{\Phi} : s \in \{0, 1, 2, ..., n\}\}$ such that $0 = b_0^{\Phi} \leq b_1^{\Phi} \leq b_2^{\Phi} \leq ... \leq b_n^{\Phi} = 1$.
Conversely consider a value Φ on $G(N)$ such that $\Phi(v) = \Phi^{Sh}(B^{\Phi} v)$ where the vector of real numbers B^{Φ} is such that $0 = b_0^{\Phi} \leq b_1^{\Phi} \leq b_2^{\Phi} \leq ... \leq b_n^{\Phi} = 1$.
Define $\alpha = \{\alpha_k : k \in \{0, 1, 2, 3, ..., n\}\}$ such that $b_s^{\Phi} = \sum_{k=1}^{s} \alpha_k$ for $s \geq 1$, $\alpha_0 = 0$. Solving for α_k, we have

$$\begin{aligned}\alpha_1 &= b_1^{\Phi} \geq 0 \\ \alpha_2 &= b_2^{\Phi} - b_1^{\Phi} \geq 0 \\ \alpha_3 &= b_3^{\Phi} - b_2^{\Phi} \geq 0 \\ &\cdots \\ \alpha_n &= b_n^{\Phi} - b_{n-1}^{\Phi} \geq 0 \\ \sum_{k=1}^{n} \alpha_k &= b_n^{\Phi} = 1\end{aligned}$$

Therefore, α is a probability distribution. Again, we have

$$\sum_{k=0}^{n} \alpha_k B^k = \left(0, \sum_{k=1}^{1} \alpha_k, \sum_{k=1}^{2} \alpha_k, ..., \sum_{k=1}^{n} \alpha_k\right) = (0, b_1^{\Phi}, b_2^{\Phi}, ..., b_n^{\Phi}) = B^{\Phi}.$$

Thus, we have

$$\Phi(v) = \Phi^{Sh}(B^{\Phi}v) = \Phi^{Sh}\Big\{\Big(\sum_{k=1}^{n}\alpha_k B^k\Big)v\Big\} = \Phi^{Sh}\Big\{\Big(\sum_{k=1}^{n}\alpha_k\Big)B^k v\Big\} = \sum_{k=1}^{n}\alpha_k\Phi^{Sh}\Big(B^k v\Big)$$
$$= \sum_{k=1}^{n}\alpha_k\Phi^{k-\mathrm{EDS}}(v) = \Phi^{\alpha-\mathrm{EDS}}(v).$$

By defining α in the above step, we have $\Phi = \Phi^{\alpha-\mathrm{EDS}}$ where $\alpha = \{\alpha_s : s \in \{0, 1, 2, ..., n\}\}$ is obtained from the transformation $B^{\Phi} \mapsto \alpha$ such that $\alpha_0 = b_0^{\Phi} = 0$, $\alpha_s = b_s^{\Phi} - b_{s-1}^{\Phi}$, $\forall s \in \{1, 2, 3, ..., n\}$. □

Remark 1. In view of observation 1(d) and linearity of the Procedural values with respect to the procedures over $k \in \{1, 2, ..., n\}$, we can easily obtain procedures for generating α-EDS values. Further, the conditions on b_v^{Φ} s follow immediately from Lemma 2[4] in [23] and observation 1(d) again. However, along the lines of [3], we have opted to provide a more detailed proof here to illustrate the relations among the coefficients.

4 Characterization

In this section we characterize $\Phi^{k-\mathrm{EDS}}$ using axioms similar to those of the Shapley value and the Equal Division rule. We first define a k-nullifying null player who nullifies the contributions of small coalitions and becomes a null player in large groups. Formally, we have the following definition.

Definition 6. Let $k \in \{1, 2, ..., n\}$ be given. Player $i \in N$ is a k-nullifying null player for $v \in G(N)$ if $v(S \cup i) = 0$ for $S \subseteq N \setminus i$ when $s < k$ and $v(S \cup i) = v(S)$ for $S \subseteq N \setminus i$ when $s \geq k$.

Axiom 11. k-nullifying null player property: (k-NNPP): For every $v \in G(N)$, it holds that $\Phi_i(v) = 0$ whenever $i \in N$ is a k-nullifying null player for v.

Note that k-NNPP requires that for a fixed k, a player that influences the contributions of the coalitions of size not more than k and becomes non-productive in coalitions of size larger than k should be rewarded zero payoff. Replacing the NP with the k-NNPP property in the characterization of the Shapley value we obtain the characterization of the k-EDS value. In the following, we provide our characterization theorem. Unless specified, we keep k fixed here.

Let us introduce two subspaces of $G(N)$ as follows:

$$G_{<k}(N) = \{v \in G(N) : v(S) = 0 \text{ for all } s \geq k\} \text{ and } G_{\geq k}(N) = \{v \in G(N) : v(S) = 0 \text{ for all } s < k\}.$$

Then using the standard notation for the direct sum of linear spaces, we get $G(N) = G_{<k}(N) \oplus G_{\geq k}(N)$. It follows that every game $v \in G(N)$ can be written as $v = v_{<k} + v_{\geq k}$ where $v_{<k} \in G_{<k}(N)$, $v_{\geq k} \in G_{\geq k}(N)$ such that $v_{<k}(S) = v(S)$ for all $s < k$, $v_{<k}(S) = 0$ for all $s \geq k$ and $v_{\geq k}(S)$ is given by 3.6.

In what follows next, we define a basis for the class $G(N)$ which will help us in showing the uniqueness of the k-EDS value latter. Consider a set of games $W = \{w_S : S \subseteq N, S \neq \emptyset\}$ where each $w_S \in G(N)$ is defined as follows.

$$w_S(T) = \begin{cases} e_S(T), & \text{if } s < k \\ u_S(T), & \text{if } s \geq k \end{cases} \tag{4.1}$$

where e_S and u_S are given by Eq.(2.1) and Eq.(2.2). The next three lemmas will be used to prove Theorem 1 that characterize the k-EDS value.

[4]Lemma 2 in [23] states that for any procedural value ψ^{γ} determined by a procedure $\gamma = (s_1, s_2, \cdots, s_n)$, the coefficients $b_1^{\Phi}, b_2^{\Phi}, \cdots, b_n^{\Phi}$ are obtained from $b_n^{\Phi} = 1$, $b_t^{\Phi} = \frac{s_{t+1}}{\binom{n}{t}}$ for $t < n$.

Lemma 1. The set of games $W = \{w_S : S \subseteq N, S \neq \emptyset\}$ is a basis for $G(N)$. Every game $v \in G(N)$ can be written as $v = \sum_{S \neq \emptyset} \lambda_S w_S$ where $\lambda_S = \sum_{T \subseteq S : t \geq k} (-1)^{s-t} v(T)$ for $s \geq k$ and $\lambda_S = v(S)$ for $s < k$.

Proof. Let $d = 2^n - 1$. Thus, the dimension of $G(N)$ is d. Let S_1, S_2, ..., S_d be a fixed sequence of all non empty subsets of N such that $1 = s_1 \leq s_2 \leq ... \leq s_d = n$ where $|S_t| = s_t$ for $t \in \{1, 2, \cdots, d\}$. Let $A = [a_{ij}]$ be a $d \times d$ matrix defined by $a_{ij} = w_{S_i}(S_j), i, j = 1, 2, 3, ..., d$. Now, if $i > j$, then $S_i \setminus S_j \neq \emptyset$ and so $u_{S_i}(S_j) = e_{S_i}(S_j) = 0$. If $i = j$, then obviously $u_{S_i}(S_j) = e_{S_i}(S_j) = 1$, thus the matrix $A = [a_{ij}]$ is an upper triangular matrix with diagonal entries 1 meaning $det(A) = 1$. Therefore, $(w_{S_i})_{i=1,2,...,d}$ constitutes a set of d independent vectors in $G(N)$. Since any linearly independent set containing d vectors form a basis of $G(N)$, therefore $W = \{w_{S_i} | i = 1, 2, 3, ..., d\}$ forms a basis for $G(N)$. Note that every element $v \in G(N)$ can be written as $v = \sum_{S \neq \emptyset} \lambda_S w_S$. For $s < k$, we have $\lambda_S = \sum_{S \neq \emptyset} \lambda_S w_S(S) = v(S)$. For $s \geq k$, $v(S) = \sum_{T \neq \emptyset, t \geq k} \lambda_T u_T(S)$. Thus, $\lambda_S = \sum_{T \subseteq S : t \geq k} (-1)^{s-t} v(T)$ for $s \geq k$. ☐

Lemma 2. If a value $\Phi : G(N) \to \mathbb{R}^n$ satisfies Eff, Sym and k-NNPP then $\Phi(w_S)$ is uniquely determined by

$$\Phi_i(w_S) = \begin{cases} 0 & \text{if } i \notin S \text{ or } s < k \\ \dfrac{1}{s} & \text{if } i \in S, s \geq k \end{cases} \tag{4.2}$$

Proof. For $i \notin S$, $w_S(T \cup i) = e_S(T \cup i) = 0$ whenever $T \subseteq N \setminus i$, $s < k$. If $s \geq k$ then $w_S(T \cup i) = u_S(T \cup i) = u_S(T) = w_S(T)$ whenever $T \subseteq N \setminus i$. Thus each $i \notin S$ is a k-nullifying null player in w_S. By k-NNPP property, $\Phi_i(w_S) = 0$ for $i \notin S$. By Symmetry, each player in S gets identical payoffs with respect to w_S. By efficiency and symmetry, $\Phi_i(w_S) = \frac{1}{s}$ for $i \in S, s \geq k$ and $\Phi_i(w_S) = 0$ for $i \in S, s < k$. ☐

Lemma 3. The k-EDS value satisfies Eff, Sym, Lin and k-NNPP.

Proof. It can be easily shown that the k-EDS value given by Eq.(3.5) satisfies Sym and Lin. We show that it also satisfies Eff and k-NNPP. From observation 1(b), we have by using Eff of the Shapley value Φ_i^{Sh} for the game $v_{\geq k}$.

$$\sum_{i \in N} \Phi_i^{k-\text{EDS}}(v) = \sum_{i \in N} \Phi_i^{Sh}(v_{\geq k}) = v(N).$$

Thus, $\Phi^{k-\text{EDS}}$ satisfies Eff. Next, we show that $\Phi^{k-\text{EDS}}$ satisfies k-NNPP. Let $k \in \{1, ..., n\}$ be fixed and $i \in N$ be a k-nullifying null player for the game $v \in G(N)$. It follows that, for $S \subset N \setminus i$ such that $s = k-1$, $v(S \cup i) = 0$. Further, for each $S \subseteq N \setminus i$, such that $s \geq k$, $v(S \cup i) = v(S)$. From 3.5, therefore, we immediately obtain that $\Phi_i^{k-\text{EDS}}(v) = 0$. ☐

Now, we are ready for the characterization theorem of the k-EDS value which goes as follows.

Theorem 1. The following statements are equivalent:

(1) Φ satisfies Eff, Sym, Lin and k-NNPP.

(2) Φ is given by $\Phi_i(v) = \sum_{S : i \in S} \frac{\lambda_S^k(v)}{s}$

(3) $\Phi(v) = \Phi^{\text{ED}}(v_{<k}) + \Phi^{\text{Sh}}(v_{\geq k})$.

(4) $\Phi = \Phi^{k-\text{EDS}}$.

Proof. By Lin, Φ is unique if it is unique on a basis. By Eff, Sym, and k-NNPP, Φ is unique on $W = \{w_S : S \subseteq N, S \neq \emptyset\}$ due to Lemma 2. In view of Lemma 1 and Lemma 2, if Φ satisfies Eff, Sym, Lin and k-NNPP then it is given by $\Phi_i(v) = \sum_{S:i\in S} \frac{\lambda_S^k(v)}{s}$. Next, we show that $\Phi_i(v) = \sum_{S:i\in S} \frac{\lambda_S^k(v)}{s}$ satisfies Eff, Sym, Lin and k-NNPP. Sym and Lin are trivial and therefore omitted.
It follows from Observation 1(b) that $\Phi_i(v) = \sum_{S:i\in S} \frac{\lambda_S^k(v)}{s} = \sum_{\substack{S:i\in S\\ s\geq k}} \frac{\lambda_S^k(v_{\geq k})}{s} = \Phi_i^{Sh}(v_{\geq k})$. By Eff of the Shapley value, $v(N) = v_{k\geq k}(N) = \sum_{i\in N} \Phi_i^{Sh}(v_{\geq k}) = \sum_{i\in N} \Phi_i(v)$. Thus, Φ is efficient. Now, if $i \in N$ is a k-nullfying null player for $v \in G(N)$, it is a null player for $v_{\geq k}$. Thus, by the null player property of the Shapley value, we have $\Phi_i(v) = \sum_{S:i\in S} \frac{\lambda_S^k(v)}{s} = \sum_{\substack{S:i\in S\\ s\geq k}} \frac{\lambda_S^k(v_{\geq k})}{s} = \Phi_i^{Sh}(v_{\geq k}) = 0$. This establishes (1)$\Leftrightarrow$(2).
It is obvious that (3)$\Leftrightarrow$(4). Moreover, by Lemma 3, $\Phi^{k-\mathrm{EDS}}$ satisfies Eff, Sym, Lin and k-NNPP. As shown at the beginning of this proof, Φ is uniquely determined by these axioms on $G(N)$. Therefore, we must have $\Phi = \Phi^{k-\mathrm{EDS}}$. It follows that (1)$\Leftrightarrow$(4). This completes the proof.[5] □

Remark 2. The logical independence of the axioms in Theorem 1 can be seen in the following examples.
Dropping Eff: The value $\Phi^1 : G(N) \to \mathbb{R}^n$ given by $\Phi_i^1(v) = \frac{1}{2^{n-1}} \sum_{S\subseteq N:s\geq k} \left\{ v(S\cup i) - v(S) \right\}$ satisfies Sym, Lin and k-NNPP but does not satisfy Eff.
Dropping Lin: Let v^* denote the dual game of $v \in G(N)$ given by, $\forall\, S \subseteq N, v^*(S) = v(N) - v(N \setminus S)$. We define a value similar to [3] that does not satisfy Lin. The Equal Surplus Division (ESD) value due to [19] is given by

$$\forall\, i \in N,\;\; \mathrm{ESD}_i(v) = v(i) + \frac{1}{n}\Big(v(N) - \sum_{j\in N} v(j)\Big).$$

Define a new value $\Phi^2 : G(N) \to \mathbb{R}^n$ given by

$$\Phi_i^2(v) = \begin{cases} \mathrm{ESD}(v^*) & \text{if } v^*(i) \neq 0 \;\forall i \in N \\ \Phi^{k-\mathrm{EDS}}(v) & \text{otherwise} \end{cases}$$

Then Φ^2 satisfies Sym, Eff and k-NNPP but does not satisfy Lin.
Dropping k-NNPP: The Solidarity value $\Phi^3 : G(N) \to \mathbb{R}^n$ due to [24] given by

$$\Phi_i^3(v) = \sum_{T\ni i} \frac{(n-t)!(t-1)!}{n!} A^v(T)$$

where $A^v(T) = \frac{1}{t}\sum_{k\in T}[v(T) - v(T \setminus k)]$ satisfies Sym, Lin, Eff but does not satisfy k-NNPP for any $k \leq n$.
Dropping Sym: Consider the basis $W = \{w_S : S \neq \emptyset\}$ for $G(N)$. Each $i \notin S$ is a k-nullifying null player for the game w_S. Let $\pi : S \to S$ be a permutation on S. Define $r = \min\{\pi(j) | j \in S\}$. Define a linear value $\Phi^4 : G(N) \to \mathbb{R}^n$ given by $\Phi_r^4(w_S) = w_S(N)$ and $\Phi_j^4(w_S) = 0$ for all $j \in N \setminus r$. Then Φ^4 satisfies Lin, Eff and k-NNPP but does not satisfy Sym.

Remark 3. Note that Eff, Sym, ADD and NP imply Lin so that in the characterization of the Shapley value, Lin can be replaced by ADD keeping the other axioms as it is. Similarly, it is easy to show from observation 1(b) that Eff, Sym, ADD and k-NNPP imply Lin and therefore, in Theorem 1, we can replace Lin by ADD.

[5] Alternatively, as suggested by an anonymous reviewer, this can also be proved (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) in a simpler way. However, the same deductions will be repeated for each implication separately, and therefore, we presented in the proof in a different format which we believe can provide more insights.

Remark 4. In [29], the Shapley value is characterized using the axioms of Eff, NP and F. It is easy to obtain a a characterization of the k-EDS value by replacing NP by k-NNPP along the same line. Indeed this follows directly from observation 1(b). Further, the Shapley value is also characterized by [30] using the axioms Eff, Sym and CSE. Axiom CSE combines ADD and NP and states that the payoff of a player from any game does not change when another game in which she is a null player is added to it. Observe that the payoff of a player from any game given by the k-EDS value does not change when another game in which she is a k-nullifying null player is added to this game. Call the corresponding axiom: k-CSE. Therefore, the k-EDS value can also be characterized using the axioms Eff, Sym and k-CSE. In [32], the elegant notions of Marginality (M) and Strong Monotonicity (SMon) are introduced in the characterization of the Shapley value. In a similar manner, considering the consolidation of marginalism and egalitarianism based on the size k of coalitions, we can define the axioms: k-Partial Marginality (k-PM) and k-Partial Monotonicity (k-PMon) as follows.

Axiom 12. k-Partial Monotonicity (k-PMon): Given two games $v, w \in G(N)$, the value $\Phi : G(N) \to \mathbb{R}^n$ satisfies k-partial monotonicity namely, for each $i \in N$, $\Phi_i(v) \geq \Phi_i(w)$ whenever either of the following holds:

(i) $m_i^v(S) \geq m_i^w(S)$ for all $S \subset N$ such that $s \geq k$ with $i \notin S$.

(ii) $v(S) \geq w(S)$ for all $S \subset N$ such that $s < k$ with $i \in S$.

Axiom 13. k-Partial Marginality (k-PM): Given two games $v, w \in G(N)$, the value $\Phi : G(N) \to \mathbb{R}^n$ satisfies k-partial marginality namely, for each $i \in N$, $\Phi_i(v) = \Phi_i(w)$ whenever either of the following holds:

(i) $m_i^v(S) = m_i^w(S)$ for all $S \subset N$ such that $s \geq k$ with $i \notin S$.

(ii) $v(S) = w(S)$ for all $S \subset N$ such that $s < k$ with $i \in S$.

In [10, 17, 30], it is shown that M (and similarly, SMon) imply CSE. In a similar manner, we can straightway show that k-PM (and similarly k-PMon) imply k-CSE. This leads to another characterization of the k-EDS value using k-PM and k-PMon as follows.

Theorem 2. A value $\Phi : G(N) \to \mathbb{R}^n$ is equal to the k-EDS value if and only if it satisfies Eff, Sym and k-PM (or k-PMon).

For a comprehensive details of all these above mentioned characterizations, we refer to our working paper [16].

5 Comparison with existing solutions

The k-EDS value is closely related to the class $\mathbf{Sol}_N$ of solidarity values proposed in [3]. Now, we discuss their relations and differences in details. For an integer p in $\{0, 1, 2, ..., n-1\}$, the payoff to player $i \in N$ given by Sol^p due to [3] has the following form.

$$\mathrm{Sol}_i^p(v) = \sum_{S \subseteq N: i \in S, s \leq p} \frac{(n-s)!(s-1)!}{n!} \Big(v(S) - v(S \backslash i) \Big) + \sum_{S \subseteq N: i \notin S, s = p} \frac{(n-s-1)!s!}{n!} \Big(v(N) - v(S) \Big). \quad (5.1)$$

If p is drawn from $\{0, 1, 2, ..., n-1\}$ according to the probability distribution $\beta = \{\beta_p : p \in \{0, 1, 2, ..., n-1\}\}$ then the solidarity value Sol^α due to [3] induced by the probability distribution β is defined as follows:

$$\mathrm{Sol}^\alpha(v) = \sum_{p=0}^{n-1} \beta_p \mathrm{Sol}^p(v) \quad (5.2)$$

We next show that for $k = n - p$, for $p \in \{0, 1, 2, ..., n-1\}$, then $\Phi^{k-\text{EDS}}(v) = \text{Sol}^p(v^*)$ where v^* is the dual game of v given by $v^*(S) = v(N) - v(N \setminus S)\ \ \forall S \subseteq N$. We have from 5.1 for each $i \in N$,

$$\begin{aligned}
\text{Sol}^p{}_i(v^*) = & \sum_{\substack{S \subseteq N: i \in S \\ s \leq p}} \frac{(n-s)!(s-1)!}{n!} \Big(v^*(S) - v^*(S \setminus i) \Big) + \sum_{\substack{S \subseteq N: i \notin S \\ s = p}} \frac{(n-s-1)!s!}{n!} \Big(v^*(N) - v^*(S) \Big) \\
= & \sum_{\substack{S \subseteq N: i \in S \\ s \leq p}} \frac{(n-s)!(s-1)!}{n!} \Big[\Big(v(N) - v(N \setminus S) \Big) - \Big(v(N \setminus i) - v(N \setminus S \cup i) \Big) \Big] \\
& + \sum_{\substack{S \subseteq N: i \notin S \\ s = p}} \frac{(n-s-1)!s!}{n!} \Big(v(N) - v(N) + v(N \setminus S) \Big) \\
= & \sum_{\substack{S \subseteq N: i \in S \\ s \leq n-k}} \frac{(n-s)!(s-1)!}{n!} \Big[\Big(v(N) - v(N \setminus i) \Big) - \Big(v(N \setminus S) - v(N \setminus S \cup i) \Big) \Big] \\
& + \sum_{\substack{S \subseteq N: i \notin S \\ s = n-k}} \frac{(n-s-1)!s!}{n!} \Big(v(N \setminus S) \Big) \\
= & \sum_{\substack{T \subseteq N: i \notin T \\ t \geq n-k}} \frac{(n-t-1)!(t)!}{n!} \Big(v(T \cup i) - v(T) \Big) + \sum_{\substack{T \subseteq N: i \notin T \\ t = k}} \frac{(n-k)!k-1!}{n!} \Big(v(T \cup i) \Big)
\end{aligned}$$

Thus, $\text{Sol}^p{}_i(v^*) = \Phi_i^{k-\text{EDS}}(v)$ whenever $p = n - k$. Here, we take $T = N \setminus S$ in the last step. It follows that, $\Phi^{k-\text{EDS}}$ and Sol^p represent two opposite social situations. In case of Sol^p, each player entering at position $\pi(i) \leq p$ obtains her contribution $v(P(\pi, i) \cup i) - v(P(\pi, i))$ upon entering, while each player entering at position $\pi(i) > p$ obtains an equal share of the remaining worth $v(N) - v(P(\pi, \pi^{-1}(p)))$. The k-EDS value $\Phi^{k-\text{EDS}}$ on the other hand, awards equal share to each player entering at position $\pi(i) \leq k$ and her marginal contributions when she enters at the position $\pi(i) > k$.

One of the key axioms in both [3] and our model to characterize the values Sol^p and $\Phi^{k-\text{EDS}}$ involves a type of null player. It is the p-null player for Sol^p and the k-nullifying null player for $\Phi^{k-\text{EDS}}$.

Given $p \in \{1, 2, 3, ..., n-1\}$, $v \in G(N)$, a player $i \in N$ is called p-null player in v if

$$\forall S \subseteq N, i \in S, s \leq p, v(S) = v(S \setminus i) \quad \text{and} \quad \forall S \subseteq N, i \notin S, s = p, v(N) = v(S).$$

A value Φ satisfies p-null player axiom if for each p-null player in v, it holds that $\Phi_i(v) = 0$. The two players, the k-nullifying null player and the p-null player are similar, but they build on two completely different social narratives. Unlike the k-nullifying null player, the p-null player is non-productive in all coalitions till they reach a size p, and the worth of all coalitions of size p where she is not a member is equal to the worth of the grand coalition. This axiom may possibly be considered somewhat demanding and technical and is particularly specific to the formulation of the solidarity value Sol^p. For instance, it might be difficult to identify situations where a player can influence a coalition of size p to generate the same worth as the grand coalition without being a part of it. The k-nullifying null player on the other hand divides the class of coalitions into two groups, in one group it acts as a nullifying player and in the other, as a null player. When $k = 1$ it is the null player and for $k = n$ it is the nullifying player. Note that the standard characterization of the Shapley value and the ED requires the null player property [28] and the nullifying player property [30] respectively. Thus it also supports our intuition that the k-nullifying null player inherits characteristics from both null and nullifying types of players.

In the following we recall Propositions 6, 8 and 11 from [3].

Proposition 3. *(Proposition 6 of [3], p. 72).* Fix any $p = \{0, ..., n-1\}$. If $p = 0$ then $\text{Sol}^0 = ED$ and if $p = n-1$ then $\text{Sol}^{n-1} = \Phi^{Sh}$.

Proposition 4. *(Proposition 8 of [3], p. 73).* A value Φ on $G(N)$ belongs to $\mathbf{Sol}_N$ if and only if it can be represented by

$$\Phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} \Big\{ b^{\Phi}_{s+1} v(S \cup i) - b^{\Phi}_s v(S) \Big\}$$

with constants $B^{\Phi} = \{b^{\Phi}_s : s \in \{0, 1, 2, ..., n\}\}$ such that

$$b^{\Phi}_0 = 0, b^{\Phi}_n = 1, \quad \texttt{and} \quad \forall s \in \{1, 2, ..., n-1\}, \quad 1 \geq b^{\Phi}_1 \geq b^{\Phi}_2 \geq ... \geq b^{\Phi}_{n-1} \geq 0.$$

Furthermore, $\Phi = \mathrm{Sol}^{\beta}$, where $\beta = \{\beta_s : s \in \{0, 1, 2, ..., n-1\}\}$ is obtained from the transformation $B^{\Phi} \mapsto \beta$ such that

$$\beta_0 = 1 - b^{\Phi}_1, \beta_{n-1} = b^{\Phi}_{n-1} \quad \text{and} \quad \forall s \in \{1, 2, ..., n-2\}, \beta_s = b^{\Phi}_s - b^{\Phi}_{s+1}.$$

Proposition 5. *(Proposition 11 of [3], p.* 80.*)* A value Φ on $G(N)$ is equal to Sol^p for $p \in \{1, 2, ..., n-1\}$ if and only if it satisfies Eff, Equal treatment of equals, Additivity and the p-null player axiom.

Proposition 3 above, shows that $\mathrm{Sol}^0 = \Phi^{ED}$ and $\mathrm{Sol}^{n-1} = \Phi^{Sh}$ and Proposition 5 characterizes Sol^p, $p \in \{1, 2, ..., n-1\}$ using Eff, Equal treatment of equals (Sym in our terminology), ADD and the p-null player axiom. However, in view of Proposition 3, the axioms in Proposition 5 cannot characterize the ED as Sol^p is equal to the ED, only when $p = 0$. Therefore, the characterization due to Proposition 5 seems quite specific making it hard to interpret it as a complete characterization from the Shapley value to the ED. On the other hand, we have a complete characterization of the k-EDS value in the range starting from $k = 1$ to $k = n$ including the Shapley value and the ED at the two extremes.

Remark 5. In [3], it is shown that the the solidarity value Φ^{sol} due to [24] (refer to Φ_3 in remark 2) belongs to the class $\mathbf{Sol}_N$. In follows from Proposition 1 that Φ^{sol} is an ESL value. In [25] (Corollary 1) the associated constants referred by Proposition 1 for Φ^{Sol} are obtained as $b^{\Phi^{Sol}}_0 = 0$, $b^{\Phi^{Sol}}_n = 1$ and $b^{\Phi^{Sol}}_s = \dfrac{1}{s+1}$ for all $s \in \{1, 2, \cdots, n-1\}$. Thus, the associated constants for Φ^{Sol} satisfy $1 = b^{\Phi^{Sol}}_n \geq b^{\Phi^{Sol}}_1 \geq b^{\Phi^{Sol}}_2 \geq \cdots \geq b^{\Phi^{Sol}}_{n-1} \geq 0 = b^{\Phi^{Sol}}_0$. On the other hand, Proposition 2 ensures that the associated constants for $\Phi^{\alpha-\mathrm{EDS}}$ satisfy $1 = b^{\Phi^{\alpha-\mathrm{EDS}}}_n \geq b^{\Phi^{\alpha-\mathrm{EDS}}}_{n-1} \geq \cdots \geq b^{\Phi^{\alpha-\mathrm{EDS}}}_1 \geq b^{\Phi^{\alpha-\mathrm{EDS}}}_0 = 0$. Therefore, Φ^{Sol} is not an α-EDS value.

6 The α-EDS value and implementation

Recall that in 3.8 we generalized the k-EDS value to include all possible values of $k \in N$ induced by the probability distribution $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ in the line of [3] and defined the α-EDS value for $v \in G(N)$ as the expected payoff given by $\Phi^{\alpha-\mathrm{EDS}}(v) = \sum_{k=1}^{n} \alpha_k \Phi^{k-\mathrm{EDS}}(v)$. Thus, the α-EDS value computes the expected payoff of each player under the probability distribution α. It follows from the above equation, that if $\alpha_1 = 1$, $\alpha_k = 0$ for $k \geq 2$ then $\Phi^{\alpha-\mathrm{EDS}} = \Phi^{Sh}$ and if $\alpha_n = 1$, $\alpha_k = 0$ for $1 \leq k \leq n-1$ then $\Phi^{\alpha-\mathrm{EDS}} = \Phi^{ED}$.

Now we propose a mechanism that implements the k-EDS and the α-EDS values for zero-monotonic games as a Subgame Perfect Nash Equilibrium (SPNE). The mechanism consists of four stages. In the first stage, each player places a bid on positions and in the second stage they place a bid on permutations. The positions may be weighted differently by the designer which is captured by α_p, but the permutations are weighted equally. These two stages determine which player will be the first proposer and what will be the nature of bargaining in the third (proposal) stage. The player after a highest bid position $k+1$ in a highest bid permutation becomes the first proposer. She proposes an allocation for all the players. If this proposal is unanimously accepted, the net amount of the bid is allocated to the players as final payoffs. Otherwise, the players in the positions less than $k+1$ leave the game with null payments and the player in position $k+1$ becomes the new proposer. The process continues till a proposal is accepted. This mechanism is an adaptation of the mechanism provided in [3] and hence we do not provide a proof for it here.

Mechanism (M)

Consider any TU-game $v \in G(N)$ and a probability distribution $\alpha = (\alpha_k)_{k=1}^n$ on N and define $A = \{i \in N : \alpha_i \neq 0\}$ to be the support of α.

Stage 1: Each player $i \in N$ makes bids $h_k^i \in \mathbb{R}$, one for each position $k \in A$, under the constraint $\sum_{k\in A} \alpha_k h_k^i = 0$. For each position $k \in A$, define the aggregate bid H_k as: $H_k = \sum_{i\in N} h_k^i$. Denote by Ω_A the subset of positions with the highest aggregate bid.

Stage 2: Each player $i \in N$ makes bids $h_\pi^i \in \mathbb{R}$, one for each permutation $\pi \in \Pi(N)$, under the constraint: $\sum_{\pi\in\Pi(N)} \frac{1}{n!} h_\pi^i = 0$. The condition given by the above equation suggests that the designer gives each permutation π equal weights, namely $\frac{1}{n!}$. For each permutation $\pi \in \Pi(N)$, the aggregate bid, defined in a similar way as in **Stage 1**, is denoted by H_π. Finally, denote by $\Omega_{\Pi(N)}$ the subset of permutations with the highest aggregate bid.

Stage 3: Pick at random any $k \in \Omega_A$ and then any $\pi \in \Omega_{\Pi(N)}$. Together, position k and permutation π induce a sequential bargaining game $G_{k,\pi}$ whose payoffs are denoted by $(g_{k,\pi}^i)_{i\in N}$. This bargaining game contains the following steps:

(i) Player $i \in N$ in position $\pi(i) = k+1$ proposes an offer $x_j^i \in \mathbb{R}$ to every other $j \in N \setminus i$.

(ii) The players other than player i, sequentially following the order π, either accept or reject the offer. If at least one player rejects it, then the offer is rejected. Otherwise the offer is accepted.

(iii) If the offer is accepted, then the payoffs are given by: $g_{k,\pi}^i = v(N) - \sum_{j\in N\setminus i} x_j^i$, and $\forall j \in N \setminus i,\ g_{k,\pi}^j = x_j^i$.

(iv) If the offer is rejected, then each player j in position $\pi(j) \leq k+1$ leaves the bargaining procedure with a null payoff, i.e. $g_{k,\pi}^j = 0$, while all the players j in positions $\pi(j) \geq k+2$ proceed to the next round to bargain over $v(P^{\pi^{-1}(k)}(\pi))$.

(v) The new proposer is player i in position $\pi(i) = k+2$. Player i makes an offer $x_j^i \in \mathbb{R}$ to every other player j such that $\pi(j) \geq k+3$. If the offer is unanimously accepted by all the players j in position $\pi(j) \geq k+3$, then the payoffs are given by $g_{k,\pi}^i = v(P^i(\pi)) - \sum_{\pi(j)>\pi(i)} x_j^i$, and $\forall j : \pi(j) > \pi(i),\ g_{k,\pi}^j = x_j^i$. If the offer is rejected, then player i in position $\pi(i) = k+2$ leaves the bargaining procedure with a null payoff. Then, step (iv) is repeated among the players j in position $\pi(j) \geq k+3$, where the new proposer is player in position $k+3$. Step (v) is repeated until a proposal is accepted. In case, the bargaining procedure reaches the situation where the only active player i is such that $\pi(i) = n$, then her payoff in $G_{k,\pi}$ is equal to $v(i)$.

Stage 4: Rewards $(z_{k,\pi}^i)_{i\in N}$ resulting from **Stage 1, 2** and **3** in $G_{k,\pi}$ are defined as: $z_{k,\pi}^i = g_{k,\pi}^i - h_k^i - h_\pi^i + \frac{H_k+H_\pi}{n}, \forall i \in N$. That is, each player pays her bids, receives an equal share of the aggregate bids H_k and H_π plus the payoff resulting from the bargaining procedure $G_{k,\pi}$. Finally, since k and π are chosen randomly from Ω_A and $\Omega_{\Pi(N)}$, the expected payoff of each player playing **Mechanism (M)** is given by

$$\forall i \in N,\ m_i = \frac{\sum_{k\in\Omega_A} \sum_{\pi\in\Omega_{\Pi(N)}} z_{k,\pi}^i}{|\Omega_A| \times |\Omega_{\Pi(N)}|}$$

Proposition 6. Consider any zero-monotonic TU game $v \in G_N$ and a probability distribution α with support $A \subseteq \{1, \cdots, n\}$. Then, **Mechanism (M)** implements the α-EDS value in SPNE.

Proof. The proof follows the steps of the proof in [3] closely adjusting for the order of the agents in the take-it-or-leave-it step. Therefore we omit the proof. □

Remark 6. Note that, we have already mentioned that our proposed mechanism is an adaptation of the mechanism of [3], however, the difference between the two mechanisms lies in the take-it-or-leave-it part of **stage 3** of the **Mechanism (B)** described in [3]. In the step 4 of stage 3 of **Mechanism (B)** in [3], all the players in position higher than (and including) $p+1$ leave with null payoff when the offer proposed by player in position $p+1$ is not accepted by consensus. In our mechanism it is the set of players in position lower than (and including) $p+1$.[6]

7 A generalized k-EDS value

In this section, we propose a class of values for $G(N)$ that covers the class of k-EDS values and the Egalitarian Shapley values due to [21]. So far, we have proposed the notion of reconciling marginalism and egalitarianism on the basis of the size of a coalition by giving equal shares (i.e., according to the ED) to each player belonging to the coalitions of size smaller than a fixed k and marginal shares from the coalitions (i.e., according to the Shapley value) of size greater or equal to this k. Recall that the ED and the Shapley value are particular cases of the α-Egalitarian Shapley value, where for $\alpha = 0$, we have the ED and for $\alpha = 1$, we have the Shapley value. Therefore, it is natural to ask whether we can introduce two parameters α_1 and α_2 ($\alpha_1, \alpha_2 \in [0,1]$) on the basis of which each player receives her α_1-Egalitarian Shapley shares from the coalitions of size less than k and her α_2-Egalitarian Shapley shares from the coalitions of size greater than or equal to k. In this section, we obtain a value that addresses this question. Let $\boldsymbol{\alpha}= (\alpha_1, \alpha_2) \in [0,1]^2$ and $k \in \{1, 2, ..., n\}$. We interpret the components α_1 and α_2 of the parameter $\boldsymbol{\alpha}$ as the level of solidarity shown to two distinct groups of coalitions which differ in their size. Following Theorem 1, we define the generalized coalition-size dependent solidarity value as follows.

Definition 7. Given $\boldsymbol{\alpha}= (\alpha_1, \alpha_2) \in [0,1]^2$ and $k \in \{1, 2, ..., n\}$, The $(k, \boldsymbol{\alpha})$-Generalized coalition-size dependent solidarity (GCDS) value denoted by $\Phi^{(k,\boldsymbol{\alpha})-GCDS} : G(N) \to \mathbb{R}^n$ with respect to the parameters k and $\boldsymbol{\alpha}$ is given by

$$\Phi_i^{(k,\boldsymbol{\alpha})\text{-GCDS}}(v) = \Phi_i^{\alpha_1-Sh}(v_{<k}) + \Phi_i^{\alpha_2-Sh}(v_{\geq k}). \tag{7.1}$$

The expression for $\Phi_i^{(k,\boldsymbol{\alpha})\text{-GCDS}}$ in Eq.(7.1) can be simplified as follows:

$$\begin{aligned}\Phi_i^{(k,\boldsymbol{\alpha})\text{-GCDS}}(v) = \quad & \Big\{\alpha_1 \Phi_i^{ED}(v_{<k}) + (1-\alpha_1)\Phi_i^{Sh}(v_{<k})\Big\} \\ & + \Big\{\alpha_2 \Phi_i^{ED}(v_{\geq k}) + (1-\alpha_2)\Phi_i^{Sh}(v_{\geq k})\Big\}\end{aligned} \tag{7.2}$$

Remark 7. From Eq.(7.2), observe that

(a) For $\alpha_1 = 1$ and $\alpha_2 = 0$, $\Phi_i^{(k,\boldsymbol{\alpha})\text{-GCDS}} = \Phi^{k-\text{EDS}}$.

(b) For $\alpha_1 = 0$ and $\alpha_2 = 1$, $\Phi_i^{(k,\boldsymbol{\alpha})\text{-GCDS}} = \text{Sol}^k$, the solidarity value due to [3].[7]

(c) For $\alpha_1 = \alpha_2 = \alpha$ (say), $\Phi_i^{(k,\boldsymbol{\alpha})\text{-GCDS}} = \Phi^{\alpha-Sh}$, the standard Egalitarian Shapley value due to [21].

Next, we obtain a characterization of the $(k, \boldsymbol{\alpha})$-GCDS value along the line of the characterizations of the k-EDS value and the Egalitarian Shapley value due to [21]. Generalizing the α-E axiom due to [21], we propose the $(k, \boldsymbol{\alpha})$- Egalitarian axiom as follows.

[6] Note that, in our formulation, it is $k+1$.

[7] This follows from the fact that $\text{Sol}^p(v) = \Phi^{k-\text{EDS}}(v^*)$, where v^* is the dual game of v. It is easy to show that $\Phi^{Sh}(v) = \Phi^{Sh}(v^*)$, see [21] also. We use this result to obtain that $\text{Sol}^p(v) = \Phi_i^{Sh}(v^*_{\geq k}) = \Phi_i^{Sh}(v_{\geq k})$.

Axiom 14. An efficient value $\Phi : G(N) \to \mathbb{R}^n$ is $(k, \boldsymbol{\alpha})$- Egalitarian if $\Phi_i(v) = \alpha_1 \frac{v_{<k}(N)}{n} + \alpha_2 \frac{v_{\geq k}(N)}{n}$ for each k-nullifying null player.

Following [21], we observe that an efficient and $(k, \boldsymbol{\alpha})$- Egalitarian value gives the k-nullifying null player a fixed fraction of α_1 of the egalitarian share from the game $v_{<k}$ plus a fixed fraction of α_2 of the egalitarian share from the game $v_{\geq k}$. When efficiency is not required, this egalitarian share is called the per capita income, see [21]. Thus, the $(k, \boldsymbol{\alpha})$-Egalitarian value exhibits solidarity for the non-productive players in a game.

Next, define the set $W^* = \{w_S^* \in G(N) | S \subseteq N,\ S \neq \emptyset\}$ with respect to the parameters k and $\boldsymbol{\alpha}$ where for each $\emptyset \neq S \subseteq N$, the TU game w_S^* is defined as follows:

$$w_S^*(T) = \begin{cases} \alpha_1 e_S(T) + (1 - \alpha_1) u_S(T) & \text{if } s < k \\ \alpha_2 e_S(T) + (1 - \alpha_2) u_S(T) & \text{if } s \geq k \end{cases}$$

It follows that $W^* = \{w_S^* \in G(N) | S \subseteq N,\ S \neq \emptyset\}$ is a basis for $G(N)$ and therefore, every $v \in G(N)$ has a unique representation of the form

$$v = \sum_{S \neq \emptyset} \lambda_S^{(k,\boldsymbol{\alpha})}(v) w_S^* \tag{7.3}$$

where the coefficients $\lambda_S^{(k,\boldsymbol{\alpha})}$ are given recursively by,

$$\lambda_S^{(k,\boldsymbol{\alpha})}(v) = v(S) \quad \forall S \subseteq N : \ s = 1. \tag{7.4}$$

$$\lambda_S^{(k,\boldsymbol{\alpha})}(v) = v(S) - (1 - \alpha_1) \sum_{\substack{T \subsetneq S \\ t < k}} \lambda_T^{(k,\boldsymbol{\alpha})}(v) - (1 - \alpha_2) \sum_{\substack{T \subsetneq S \\ t \geq k}} \lambda_T^{(k,\boldsymbol{\alpha})}(v), \tag{7.5}$$

$$\forall S \subseteq N \ : s > 1.$$

In the following, we give an illustration of the above formulations with an example.

Example 2. Let $N = \{1, 2, 3\}$. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ be given. Also let, $k = 2$. Then using the definition of w_S^* for each $S \subseteq N$, we obtain the matrix of the basis $W^* = \{w_S^* \in G(N) | S \subseteq N,\ S \neq \emptyset\}$ as follows.

$w_S^*(T)$/T	$w_{\{1\}}^*$	$w_{\{2\}}^*$	$w_{\{3\}}^*$	$w_{\{1,2\}}^*$	$w_{\{1,3\}}^*$	$w_{\{2,3\}}^*$	$w_{\{1,2,3\}}^*$
$\{1\}$	1	0	0	0	0	0	0
$\{2\}$	0	1	0	0	0	0	0
$\{3\}$	0	0	1	0	0	0	0
$\{1,2\}$	$(1-\alpha_1)$	$(1-\alpha_1)$	0	1	0	0	0
$\{1,3\}$	$(1-\alpha_1)$	0	$(1-\alpha_1)$	0	1	0	0
$\{2,3\}$	0	$(1-\alpha_1)$	$(1-\alpha_1)$	0	0	1	0
$\{1,2,3\}$	$(1-\alpha_1)$	$(1-\alpha_1)$	$(1-\alpha_1)$	$(1-\alpha_2)$	$(1-\alpha_2)$	$(1-\alpha_2)$	1

Table 2: The Matrix Representation of the basis vectors w_S^* of Example 2.

It follows from Table 2 that the set $W^* = \{w_S^* \in G(N) | S \subseteq N,\ S \neq \emptyset\}$ is a basis for $G(N)$, where $N = \{1, 2, 3\}$. Every $v \in G(N)$ is expressible as a unique linear combination of these basis vectors as given by Eq.(7.3). Thus we have, for all $S \subseteq N$,

$$v(S) = \lambda_{\{1\}}^{(2,\boldsymbol{\alpha})}(v) w_{\{1\}}^*(S) + \lambda_{\{2\}}^{(2,\boldsymbol{\alpha})}(v) w_{\{2\}}^*(S) + \lambda_{\{3\}}^{(2,\boldsymbol{\alpha})}(v) w_{\{3\}}^*(S) + \lambda_{\{1,2\}}^{(2,\boldsymbol{\alpha})}(v) w_{\{1,2\}}^*(S)$$
$$+ \lambda_{\{1,3\}}^{(2,\boldsymbol{\alpha})}(v) w_{\{1,3\}}^*(S) + \lambda_{\{2,3\}}^{(2,\boldsymbol{\alpha})}(v) w_{\{2,3\}}^*(S) + \lambda_{\{1,2,3\}}^{(2,\boldsymbol{\alpha})}(v) w_{\{1,2,3\}}^*(S).$$

Using the values of w_S^* from Table 2 we obtain the following system of equations.

$$\begin{aligned}
v(1) &= \lambda_{\{1\}}^{(2,\boldsymbol{\alpha})}(v), \;\; v(2) = \lambda_{\{2\}}^{(2,\boldsymbol{\alpha})}(v), \;\; v(3) = \lambda_{\{3\}}^{(2,\boldsymbol{\alpha})}(v), \\
v(1,2) &= \lambda_{\{1\}}^{(2,\boldsymbol{\alpha})}(v)(1-\alpha_1) + \lambda_{\{2\}}^{(2,\boldsymbol{\alpha})}(v)(1-\alpha_1) + \lambda_{\{1,2\}}^{(2,\boldsymbol{\alpha})}(v) \\
v(1,3) &= \lambda_{\{1\}}^{(2,\boldsymbol{\alpha})}(v)(1-\alpha_1) + \lambda_{\{3\}}^{(2,\boldsymbol{\alpha})}(v)(1-\alpha_1) + \lambda_{\{1,3\}}^{(2,\boldsymbol{\alpha})}(v) \\
v(2,3) &= \lambda_{\{2\}}^{(2,\boldsymbol{\alpha})}(v)(1-\alpha_1) + \lambda_{\{3\}}^{(2,\boldsymbol{\alpha})}(v)(1-\alpha_1) + \lambda_{\{2,3\}}^{(2,\boldsymbol{\alpha})}(v) \\
v(1,2,3) &= \lambda_{\{1\}}^{(2,\boldsymbol{\alpha})}(v)(1-\alpha_1) + \lambda_{\{2\}}^{(2,\boldsymbol{\alpha})}(v)(1-\alpha_1) + \lambda_{\{3\}}^{(2,\boldsymbol{\alpha})}(v)(1-\alpha_1) + \lambda_{\{1,2\}}^{(2,\boldsymbol{\alpha})}(v)(1-\alpha_2) \\
&\quad + \lambda_{\{1,3\}}^{(2,\boldsymbol{\alpha})}(v)(1-\alpha_2) + \lambda_{\{2,3\}}^{(2,\boldsymbol{\alpha})}(v)(1-\alpha_2) + \lambda_{\{1,2,3\}}^{(2,\boldsymbol{\alpha})}(v)
\end{aligned}$$

Solving this system of equations, we get,

$$\begin{aligned}
\lambda_{\{1\}}^{(2,\boldsymbol{\alpha})}(v) &= v(1), \;\; \lambda_{\{2\}}^{(2,\boldsymbol{\alpha})}(v) = v(2), \;\; \lambda_{\{3\}}^{(2,\boldsymbol{\alpha})}(v) = v(3), \\
\lambda_{\{1,2\}}^{(2,\boldsymbol{\alpha})}(v) &= v(1,2) - v(1)(1-\alpha_1) - v(2)(1-\alpha_1), \\
\lambda_{\{1,3\}}^{(2,\boldsymbol{\alpha})}(v) &= v(1,3) - v(1)(1-\alpha_1) + v(3)(1-\alpha_1) \\
\lambda_{\{2,3\}}^{(2,\boldsymbol{\alpha})}(v) &= v(2,3) - v(2)(1-\alpha_1) - v(3)(1-\alpha_1) \\
\lambda_{\{1,2,3\}}^{(2,\boldsymbol{\alpha})}(v) &= v(1,2,3) - v(1)(1-\alpha_1) - v(2)(1-\alpha_1) - v(3)(1-\alpha_1) \\
&\quad -(1-\alpha_2)\Big\{v(1,2) - v(1)(1-\alpha_1) - v(2)(1-\alpha_1)\Big\} \\
&\quad -(1-\alpha_2)\Big\{v(1,3) - v(1)(1-\alpha_1) - v(3)(1-\alpha_1)\Big\} \\
&\quad -(1-\alpha_2)\Big\{v(2,3) - v(2)(1-\alpha_1) - v(3)(1-\alpha_1)\Big\}.
\end{aligned}$$

These values are in accordance with the recursive formulae for $\lambda_S^{(k,\boldsymbol{\alpha})}(v)$ given by Eq.(7.4) and Eq.(7.5). The next theorem characterises the value $\Phi_i^{(k,\boldsymbol{\alpha})\text{-GCDS}}$.

Theorem 3. The following statements are equivalent:

(1) Φ satisfies Eff, Sym, Lin and is $(k,\boldsymbol{\alpha})$-Egalitarian.

(2) Φ is given by $\Phi_i(v) = \sum_{S:i\in S} \dfrac{\lambda_S^{(k,\boldsymbol{\alpha})}(v)}{s}$

(3) $\Phi = \Phi^{(k,\boldsymbol{\alpha})-\text{GCDS}}$

(4) $\Phi(v) = \Phi^{\alpha_1 - Sh}(v_{<k}) + \Phi^{\alpha_2 - Sh}(v_{k\geq k})$.

Proof. The proof proceeds exactly in the same way as for Theorem 1, and is therefore omitted. □

8 Conclusion

In this paper we provide a new value for TU games that exhibits the characteristics of the Equal Division rule in small coalitions and the Shapley value in sufficiently large coalitions. Our model generates a whole range of values that includes the Shapley value and the ED at its two extremes. We provide a characterization of the proposed value and have developed a mechanism that implements our values in subgame perfect Nash equilibrium. We also generalize the k-EDS value by taking all possible expectations over the threshold coalition size k. One interesting extension of this idea could be to explore games with coalition structures where partition of the players based on who constitutes a family and who does not. Then we could consider ED within a family and marginalism outside. Additionally, a further straightforward generalisation of this value can be obtained by integrating several values instead of just ED and the Shapley value for multiple threshold levels of k's where k's range from 1 to n. We leave these questions for future research.

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