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IFAC PapersOnLine 56-2 (2023) 8536–8541

A Passivity based Approach to Predefined-Time Stabilization

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Abstract: Passivity theory is an efficacious framework in designing control laws, in particular, predefined-time stabilizing control laws. To that end, this paper discusses a predefined-time variant of passivity, associated Lyapunov tools, and basic analysis of this property through feedback interconnection. In particular, a series of predefined-time passivity definitions are discussed which allow to establish a relationship between the passivity framework and predefined-time stability. These definitions are exploited for designing control laws that render a passive system predefined-time stable about an equilibrium point. Moreover, several negative feedback interconnections of these systems are discussed to investigate the predefined-time passivity properties and predefined-time stability of an equilibrium point (when external inputs are zero). The efficacy of the proposed results is illustrated through academic and realistic examples, and comparison is also done with other existing methods.

Keywords: Control design, Nonlinear Control Systems, Passivity theory, Storage functions and Stabilization

1. INTRODUCTION

Passive systems are dynamical systems that do not generate energy, but rather dissipate. In general, passivity is defined in terms of an inequality involving the storage function and the supply rate notions that reveals in essence that the stored energy is bounded by the supplied energy to the passive system. These storage functions can be further used as Lyapunov candidate functions when the external supply is zero and thus this theory is of paramount importance in the study of stability problems. One of the important features of this passivity property is that it is preserved under negative feedback interconnections. In recent years, this theory has been a powerful tool for the stability and stabilization of control systems (Ortega et al. (1998)). It has potential applications in electrical circuits, mechanical systems (Acosta (2005), Ortega et al. (2017)), multi-agent systems (Sharf and Zelazo (2019)), and chemical processes (Sira-Ramirez and Angulo-Nunez (1997)). In the absence of external supply to the system, passivity definitions reveal that the stored energy in a passive system goes to zero as the time approaches infinity. But, in various engineering and industrial applications, stored energy must go to zero in some finite/fixed or predefined time rather than the infinite duration. This observation motivates the present work. Finite-time stability of origin has been investigated by many researchers in the past Bhat and Bernstein (2000). The notion of finite-time passivity was introduced in Hou et al. (2016) and then further exploited to solve various problems such as synchronization of multi-agent systems Ren et al. (2020), neural networks Wang et al. (2017) and attitude stabilization and tracking of the rigid body Gui and Vukovich (2016). However, in all these finite-time problems, the settling time depends on initial state values, which are rather difficult to estimate accurately for the practical systems. This led to the emergence of a concept of fixed-time stability Cruz-Zavala et al. (2011). It provides uniformity with respect to the initial conditions in calculating the upper bound of the settling time, however, this upper bound explicitly depends on system parameters Angulo et al. (2013) and cannot be selected a priori. Fixed-time passivity which combines the notions of fixed-time stability and passivity theory has been investigated for several applications, in particular, memristive neural networks Wang et al. (2019). Later on, in Corradini and Cristofaro (2018), authors derived the way of computing settling time but did not discuss any strategy to design several controller parameters to ensure the desired settling time. Hence, in most of the problems, the settling time or its upper bound depends on initial conditions and system parameters.

These restrictions encouraged the research on the upper bounds of the settling time that can be chosen in advance and is independent of any initial condition and system parameters. This objective is fulfilled in the recent work on predefined-time stability developed in (Pal et al. (2020), Singh et al. (2022)). In fact, this stability notion is advantageous in comparison with finite-time and fixed-time cases in the way that the upper bound of the settling time can be chosen a priori and is independent of any initial

⋆ "B. S., N. A., and W. N. gratefully acknowledge support by the UKRI Belfast Maritime Consortium".

10.1016/j.ifacol.2023.10.013

state value. It can be noted that although there has been eminent literature on the theory and applications of passivity framework, no study has been conducted to discuss predefined-time passivity properties, and to design control methodology that renders the passive systems, predefined-time stable about an equilibrium point. This encouraged us to discuss the notion of predefined-time passivity, since time is a critical factor in all real-world control applications and passivity framework provides potential benefits such as it is independent of the state which is unrealistic in various control problems.

The main contributions of this work are as follows.

(a) We discuss a series of definitions of predefined-time passive systems as these definitions draw a considerable attention due to the elegant feature that predefined-time passivity implies predefined-time stability when external inputs are zero.
(b) Further, we derive a predefined-time passivity based control methodology which only requires the information of system output for passive systems to achieve predefined time convergence. Moreover, passivity framework has not been explored for controller design to achieve predefined time convergence in spite of the fact that many physical systems such as electrical and mechanical systems mimic the nature of the passive systems.
(c) Furthermore, feedback interconnections of these systems are discussed to investigate the predefined-time passivity properties and predefined-time stability of the equilibrium point. As a result, it provides a nice property that predefined-time passivity remains preserved when two predefined-time passive systems are connected in negative feedback connection. As a consequence, it extends a framework of predefined-time convergence for large scale interconnected systems.
(d) As a validation to the proposed results, mathematical and realistic examples are provided to illustrate the fact that passive systems achieve convergence in predefined time by the controller having only the information of the system output. In the end, comparison is also done with other existing methods.

The rest of the paper is structured as follows. The mathematical notions and preliminary results are noted in Section 2. Section 3 provides the main results. Several interconnections of predefined-time passive systems and further control design methodology is discussed in Section 4. Examples and comparison with other existing methods are provided in Section 5. Finally, conclusions end the paper.

2. PRELIMINARIES: STABILITY AND PASSIVITY

$\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>p}$ denote the sets of real numbers, non-negative real numbers and real numbers greater than p. $\mathbb{R}^n$ denotes the n dimensional Euclidean space. $\|\cdot\|$ represents the Euclidean norm. A function $\Psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if it is continuous and strictly increasing with $\Psi(0) = 0$ (Khalil (1993)).

2.1 A Brief Review on the Predefined-Time Stability

At first, we consider the time-varying first-order system

$$\dot{x} = u := \begin{cases} \dfrac{-\eta(e^x - 1)}{e^x(t_p - t)}, & \text{if } t_0 \leq t < t_p \\ 0, & t \geq t_p \end{cases} \tag{1}$$

where $x \in \mathbb{R}$, $\eta \in \mathbb{R}_{>1}$, $t_0 \in \mathbb{R}_{\geq 0}$ is the initial time and $t_p = T_P + t_0$, T_P is a predefined time duration. It is evident from (1) that u is continuous in x for all t and measurable in t for all x and also it is bounded by a quantity that is integrable over a fixed time interval (Pal et al. (2020)), thus the solution of (1) is interpreted in the sense of Carathéodory (Carathéodory (2004)) upto time t_p and in classical sense after $t > t_p$. Hence, the existence and uniqueness of solutions of (1) are established in the entire time interval $[t_0, \infty)$.

Remark 1. We note that system (1) is uniformly stable, consider some δ independent of t_0, such that $|x(t_0)| < \delta$, then the solution of (1) is given by $|x(t)| = |\ln(\mathcal{C}(t_p - t)^\eta + 1)|$, where $\mathcal{C} = \frac{e^{x(t_0)}-1}{(t_p - t_0)^\eta}$, which provides $|x(t)| \leq |x(t_0)|$. Thus, one can select $\delta = \epsilon$ for given $\epsilon > 0$.

Consider a general forced time-varying nonlinear system

$$\dot{x} = f(t, x, \gamma, u), \quad x(t_0) = x_0 \in D \subset \mathbb{R}^n \tag{2}$$

where $x \in D \subset \mathbb{R}^n$ is the system state vector, $\gamma \in \mathbb{R}^p$ represents the vector of constant system parameters, $u : u(t, x, t_p) \in \mathbb{R}^m$ is the input vector ($t_p > t_0$) and $f : \mathbb{R}_{\geq 0} \times D \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear function in x and u such that $f(t, 0, \gamma, 0) = 0$ for all $t \geq t_0$, that is, origin is an equilibrium point of (2). The solutions of (2) are interpreted as for system (1). Let $D \subset \mathbb{R}^n$ be a domain containing the origin.

Definition 1. Pal et al. (2020) (Predefined-time stability) The system (2) is predefined-time stable at the origin for a control $u : u(t, x, t_p)$ if

(a) it is asymptotically stable and any solution $x(t, t_0, x_0)$ of (2) converges to the origin at some finite time, that is, $x(t, t_0, x_0) = 0$ for all $t \geq t_0 + T(t_0, x_0)$, where $T : \mathbb{R}_{\geq 0} \times D \to \mathbb{R}_{\geq 0}$ is the convergence time,
(b) it is possible to select in advance an upper bound of the convergence time duration $T_P > 0$ ($t_p > t_0$) which is independent of any initial conditions ($x_0 \in D$), and the condition: $T_P \geq T_F$ can be established for all $x_0 \in D$, where T_F is the exact time at which the system trajectories converge to the origin.

Lemma 1. Pal et al. (2020) Let us consider the system (2). Let $\Upsilon_1(x)$ and $\Upsilon_2(x)$ be positive definite continuous functions on D. Suppose that there exist a continuously differentiable function $V : [t_0, \infty) \times D \to \mathbb{R}_{\geq 0}$ and a constant $\eta \in \mathbb{R}_{>1}$ such that

(a) $\Upsilon_1(x) \leq V(t, x) \leq \Upsilon_2(x)$, $\forall\, t \in [t_0, \infty)$, $\forall\, x \in D \setminus \{0\}$
(b) $V(t, 0) = 0$, $\forall\, t \in [t_0, \infty)$
(c) $\dot{V}(t, x) \leq \begin{cases} \dfrac{-\eta(e^{V(t,x)} - 1)}{e^{V(t,x)}(t_p - t)}, & \text{if } t_0 \leq t < t_p \\ 0, & t \geq t_p \end{cases}$

for $V \neq 0$, then the origin $x = 0$ is predefined-time stable and $t_p = T_p + t_0$, where T_p is the time interval chosen in advance (predefined).

2.2 A Brief Review on Dissipativity and Passivity

Definition 2. ($\mathcal{L}_{2e}^n$, $\mathcal{L}_2^n$ norms and $\mathcal{L}_{2e}^n$ space): The $\mathcal{L}_{2e}^n$ norm of a given signal $s : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is defined by:

$\|s(t)\|_{2T} := \left(\int_{t_0}^{T} \|s(t)\|^2 dt\right)^{\frac{1}{2}}$ and the $\mathcal{L}_2^n$ norm is defined by $\|s(t)\|_2 := \lim_{T\to\infty} \|s(t)\|_{2T}$. One can say that s belongs to $\mathcal{L}_{2e}^n$ space if and only if $\|s(t)\|_{2T} < \infty$ for all $t \geq t_0$.

Consider the general forced nonlinear time-varying system

$$\Sigma : \begin{cases} \dot{x} = g(t, x, u), & x(t_0) = x_0 \in D \subset \mathbb{R}^n \\ y = h(t, x, u) \end{cases} \tag{3}$$

where $x \in D \subset \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^m$ is the output, $g : \mathbb{R}_{\geq 0} \times D \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}_{\geq 0} \times D \times \mathbb{R}^m \to \mathbb{R}^m$ are nonlinear functions in x and u. The solutions of (3) are interpreted as for system (1). We assume that $(t, 0, 0)$ is the equilibrium point of the system (3), that is, $g(t, 0, 0) = 0$ and $h(t, 0, 0) = 0$, $\forall\ t > t_0$. We further assume that, (3) defines a causal dynamic operator $\Sigma : \mathcal{L}_{2e}^n \to \mathcal{L}_{2e}^n : u \mapsto y$.

Definition 3. Ortega et al. (1998) (Dissipativity): Σ is dissipative with respect to the supply rate $w(u, y) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ if and only if there exists a continuously differentiable positive definite storage function $V : D \to \mathbb{R}_{\geq 0}$, such that $V(x(T)) \leq V(x(t_0)) + \int_{t_0}^{T} w(u(\tau), y(\tau))d\tau$, $\forall\ u, \forall\ T \geq t_0$ and $\forall\ x(t_0) \in D \subset \mathbb{R}^n$.

Definition 4. Ortega et al. (1998) (Passivity): Σ is said to be passive if it is dissipative with supply rate $w(u, y) = u^\top y$. Σ is said to be input strictly passive (ISP) if it is dissipative with supply rate $w(u, y) = u^\top y - u^\top \phi(u)$, with $u^\top \phi(u) > 0$ for all $u \neq 0$. Moreover, Σ is output strictly passive (OSP) if it is dissipative with supply rate $w(u, y) = u^\top y - y^\top \phi(y)$, for all $y \neq 0$ and $y^\top \phi(y) > 0$. Furthermore, Σ is strictly passive or state strictly passive (SSP) if it is dissipative with supply rate $w(u, y, x) = u^\top y - W(x)$, with W as a positive definite function.

Definition 5. (Khalil (1993)) The system (3) is zero-state observable if no solution of $\dot{x} = g(t, x, 0)$ stays identically in the set $\bar{S} = \{x \in \mathbb{R}^n \mid h(t, x, 0) = 0\}$, other than $x = 0$.

3. MAIN RESULTS

Definition 6. Σ is **predefined-time output strictly passive (PTOSP)** if it is dissipative with supply rate $w(u, y) = u^\top y - y^\top \phi(y)$ for $t_0 \leq t < t_p$, with $\phi_i(y_i) := \frac{\eta_i(e^{y_i}-1)}{e^{y_i}(t_p-t)}$ for all $y := [y_1, y_2, \cdots, y_m]^\top$ and $\phi(y) := [\phi_1(y_1), \phi_2(y_2), \cdots, \phi_m(y_m)]^\top$, where $\eta_i \in \mathbb{R}_{>1}, i = 1 \cdots, m$ and $t_p = T_P + t_0$, T_P is the time duration independent of any initial condition ($x_0 \in D$) and can be chosen a priori, and OSP for $t \geq t_p$. Furthermore, Σ is **predefined-time state strictly passive (PTSSP)** if it is dissipative with supply rate $w(u, y, x) = u^\top y - W(x) - \frac{\eta \Psi(V)\left(e^{\Psi(V)}-1\right)}{e^{\Psi(V)}(t_p-t)}$ for $t_0 \leq t < t_p$, with $\eta \in \mathbb{R}_{>1}$, $W(\cdot)$ as a positive definite function and class $\mathcal{K}$ function Ψ and SSP for $t \geq t_p$.

Example (Predefined-time output strictly passive (PTOSP))

Consider the following forced nonlinear system

$$\dot{y} = \theta(y) + u; \quad y \in \mathcal{L}_{2e} \tag{4}$$

$$\theta(y) := \begin{cases} \dfrac{-\eta(e^y - 1)}{e^y(t_p - t)}, & \text{if } t_0 \leq t < t_p \\ -ay, & t \geq t_p \end{cases}$$

where $u \in \mathcal{L}_{2e}$ is the input and $y \in \mathcal{L}_{2e}$ is the output, $\eta \in \mathbb{R}_{>1}$, $a > 0$ and $t_0 \in \mathbb{R}_{\geq 0}$ is the initial time. Let us select the positive definite storage function $V(y) = \frac{1}{2}y^2$. The time derivative of the storage function along the system (4) for $t_0 \leq t < t_p$ is given by, $\dot{V} = y\dot{y} = \frac{-\eta y(e^y-1)}{e^y(t_p-t)} + uy \leq uy - y\phi(y)$, where $\phi(y) = \frac{\eta(e^y-1)}{e^y(t_p-t)}$. And, the time derivative of the storage function for $t \geq t_p$ is given by: $\dot{V} \leq uy - ay^2$ (OSP). Thus, from the Definition 6, the system (4) is PTOSP.

Example (Predefined-time state strictly passive (PTSSP))

Consider the following forced nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta(x_1) \\ \dot{x}_2 &= -x_1 + \theta(x_2) + u; \quad y = x_2, \quad y \in \mathcal{L}_{2e} \end{aligned} \tag{5}$$

$$\theta(x_i) := \begin{cases} \dfrac{-\eta_i(e^{x_i} - 1)}{e^{x_i}(t_p - t)} - ax_i, & \text{if } t_0 \leq t < t_p \\ -ax_i, & t \geq t_p \end{cases}$$

for $i = 1, 2$, where $a > 0$, $u \in \mathcal{L}_{2e}$ is the input, $y \in \mathcal{L}_{2e}$ is the output and $\eta_i \in \mathbb{R}_{>1}$. Consider the following positive definite storage function

$$V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \tag{6}$$

The time derivative of V along (5) for $t_0 \leq t < t_p$ is

$$\begin{aligned} \dot{V} &= -\frac{\eta_1 x_1(e^{x_1} - 1)}{e^{x_1}(t_p - t)} - \frac{\eta_2 x_2(e^{x_2} - 1)}{e^{x_2}(t_p - t)} - ax_1^2 - ax_2^2 + ux_2 \\ &\leq -\frac{\eta_1 |x_1|(e^{|x_1|} - 1)}{e^{|x_1|}(t_p - t)} - \frac{\eta_2 |x_2|(e^{|x_2|} - 1)}{e^{|x_2|}(t_p - t)} - a(x_1^2 + x_2^2) + uy \end{aligned}$$

From (6), it can be observed that, $\sqrt{V} \leq \max\{|x_1|, |x_2|\}$. Now, it is clear that for any time $t \in [t_0, t_p)$, the function $\max\{|x_1|, |x_2|\}$ gives either $|x_1|$ or $|x_2|$. Let us assume that firstly it gives $|x_1|$, then $\sqrt{V} \leq |x_1|$. Hence, $\dot{V}$ for $t_0 \leq t < t_p$, $\dot{V} \leq -\frac{\eta_1 \sqrt{V}(e^{\sqrt{V}}-1)}{e^{\sqrt{V}}(t_p-t)} - W(x) + uy$, where $W = a(x_1^2 + x_2^2)$ is a positive definite function. Similar analysis holds if the function $\max\{|x_1|, |x_2|\}$, yields $|x_2|$ and the final results remain the same. Further, the time derivative of the storage function for $t \geq t_p$ is given by $\dot{V} \leq uy - W(x)$ (SSP).

4. PREDEFINED-TIME PASSIVITY AND ITS CONTROL DESIGN METHODOLOGY FOR INTERCONNECTED SYSTEMS

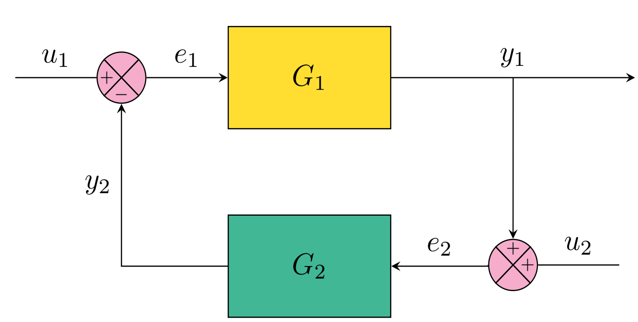

Fig. 1. Negative feedback connection of two systems G_1 and G_2

4.1 Predefined-time passivity of interconnected systems

Consider the interconnection of two systems G_1 and G_2 as shown in Fig. 1, where G_1 and G_2 are represented by (3) with $y_1, y_2 \in \mathbb{R}^m$ as the outputs, $x_1, x_2 \in D \subset \mathbb{R}^n$ as the states and $u_1, u_2 \in \mathbb{R}^m$ as the inputs respectively. We assume that origin is the equilibrium point of this interconnected system.

Theorem 1. 1) Feedback connection of two PTSSP systems is a PTSSP system. Further if external inputs u_1 and u_2 are zero, then it is predefined-time stable about the origin.

2) Feedback connection of two PTOSP systems is a PTOSP system. Further if external inputs u_1 and u_2 are zero and G_1, G_2 are zero-state observable, then it is predefined-time stable about the origin.

Proof: Suppose that V_1 and V_2 are positive definite storage functions of subsystems G_1 and G_2 respectively.

(1) For the first case, let us assume that both subsystems are PTSSP i.e., $\dot{V}_1 \le e_1^\top y_1 - W_1 - \frac{\eta_1 \Psi(V_1)\left(e^{\Psi(V_1)}-1\right)}{e^{\Psi(V_1)}(t_p-t)}$ for $t_0 \le t < t_p$, with $\eta_1 \in \mathbb{R}_{>1}$, W_1 as a positive definite function and class $\mathcal{K}$ function Ψ and $\dot{V}_1 \le e_1^\top y_1 - W_1$ for $t \ge t_p$. Similarly, $\dot{V}_2 \le e_2^\top y_2 - W_2 - \frac{\eta_2 \Psi(V_2)\left(e^{\Psi(V_2)}-1\right)}{e^{\Psi(V_2)}(t_p-t)}$ for $t_0 \le t < t_p$, with $\eta_2 \in \mathbb{R}_{>1}$, W_2 as a positive definite function and $\dot{V}_2 \le e_2^\top y_2 - W_2$ for $t \ge t_p$. Taking $V := V_1 + V_2$, it holds $V \le 2(\max\{V_1, V_2\})$ $\forall\, x_1, x_2 \in D$. Therefore $\Psi(\frac{V}{2}) \le \Psi(V_i),\ i = 1, 2$. Also, we can define $W := W_1 + W_2$ as positive definite. Thus, $\dot{V} \le e_1^\top y_1 + e_2^\top y_2 - W - \frac{\eta \Psi(\frac{V}{2})\left(e^{\Psi(\frac{V}{2})}-1\right)}{e^{\Psi(\frac{V}{2})}(t_p-t)}$ for $t_0 \le t < t_p$, since the function $\max\{V_1, V_2\}$ gives either V_1 or V_2 at a particular time instant, with $\eta = \eta_1$ or $\eta_2 \in \mathbb{R}_{>1}$, and $\dot{V} \le e_1^\top y_1 + e_2^\top y_2 - W$ for $t \ge t_p$. From Fig. 1, we can write $e_1 = u_1 - y_2$, and $e_2 = u_2 + y_1$. From $e_1^\top y_1 + e_2^\top y_2$, one can get $(u_1 - y_2)^\top y_1 + (u_2 + y_1)^\top y_2 = u_1^\top y_1 + u_2^\top y_2$ which is exactly what we needed to prove PTSSP.
Further if u_1 and u_2 are zero, then the proof is obvious from Lemma 1 because storage function becomes Lyapunov function.

(2) For the second case, assume that both subsystems are PTOSP i.e., $\dot{V}_1 \le e_1^\top y_1 - y_1^\top \phi(y_1)$ for $t_0 \le t < t_p$, with $\phi_i(y_{1i}) := \frac{\eta_{1i}(e^{y_{1i}}-1)}{e^{y_{1i}}(t_p-t)}, i = 1 \cdots, m$, for all $\phi(y_1) := [\phi_1(y_{11}), \phi_2(y_{12}), \cdots, \phi_m(y_{1m})]^\top$ and $y_1 := [y_{11}, y_{12}, \cdots, y_{1m}]^\top$, where $\eta_{1i} \in \mathbb{R}_{>1}$ and $\dot{V}_1 \le e_1^\top y_1 - y_1^\top y_1$ for $t \ge t_p$. Similarly, $\dot{V}_2 \le e_2^\top y_2 - y_2^\top \phi(y_2)$ for $t_0 \le t < t_p$, with $\phi_i(y_{2i}) := \frac{\eta_{2i}(e^{y_{2i}}-1)}{e^{y_{2i}}(t_p-t)}$ for all $y_2 := [y_{21}, y_{22}, \cdots, y_{2m}]^\top$ and $\phi(y_2) := [\phi_1(y_{21}), \phi_2(y_{22}), \cdots, \phi_m(y_{2m})]^\top$, where $\eta_{2i} \in \mathbb{R}_{>1}$ and $\dot{V}_2 \le e_2^\top y_2 - y_2^\top y_2$ for $t \ge t_p$. Taking $V := V_1 + V_2$. Thus, $\dot{V} \le e_1^\top y_1 + e_2^\top y_2 - y_1^\top \phi(y_1) - y_2^\top \phi(y_2)$ for $t_0 \le t < t_p$, and $\dot{V} \le e_1^\top y_1 + e_2^\top y_2 - y_1^\top y_1 - y_2^\top y_2$ for $t \ge t_p$. Again, on evaluating the expression $e_1^\top y_1 + e_2^\top y_2$, we get what we needed to prove PTOSP. Further if u_1 and u_2 are zero, then $\dot{V} \le -y_1^\top \phi(y_1) - y_2^\top \phi(y_2)$ where $y_i^\top \phi(y_i) > 0$ for all $y_i \ne 0$. Thus, $\dot{V} = 0$ in time $t \le t_p$ if and only if $y_1 = 0$ and $y_2 = 0$ in time $t \le t_p$. Note that $y_2 = 0 \implies e_1 = 0$, in time $t \le t_p$. Since, G_1 is zero-state observable, then, $y_1 = 0 \implies x_1 = 0$ in time $t \le t_p$. In a similar way, $y_1 = 0 \implies e_2 = 0$ in time $t \le t_p$ and due to zero-state observability of G_2, $y_2 = 0 \implies x_2 = 0$ in time $t \le t_p$. Hence, the origin of the interconnected system is predefined-time stable in time $t \le t_p$. After $t \ge t_p$, $\dot{V} \le -y_1^\top y_1 - y_2^\top y_2$, infers that the origin is asymptotically stable (i.e., trajectories remain at the origin) using invariance rule (Khalil (1993)) and zero-state observability. ■

4.2 Predefined-time passivity based control

Consider the interconnection of G_1 and G_2 as shown in Fig. 1, where G_1 is represented by (3).

Theorem 2. Suppose that G_1 and G_2 satisfy either of the following assumptions:

(1) G_1 is PTSSP and G_2 is a linear map $u = -ky$, with $k > 0$ or a static non-linearity, $u = -\varphi(y)$, with sector condition $y^\top \varphi(y) > 0$ for all $y := [y_1, y_2, \cdots, y_m]^\top \ne 0$, where $\varphi(y) = [\varphi_1(y_1), \varphi_2(y_2), \cdots, \varphi_m(y_m)]^\top$.

(2) G_1 is passive and zero-state observable and G_2 is a nonlinear map $u = -\phi(y) - My$, with $\phi(y) = [\phi_1(y_1), \phi_2(y_2), \cdots, \phi_m(y_m)]^\top$, and $\phi_i(y_i) = \frac{\eta_i(e^{y_i}-1)}{e^{y_i}(t_p-t)}$ for $t_0 \le t < t_p$, and $\phi_i(y_i) = 0$ for $t \ge t_p$, where $\eta_i \in \mathbb{R}_{>1}$, $M \in \mathbb{R}^{m\times m}$ is a positive matrix and $y := [y_1, y_2, \cdots, y_m]^\top$.

Then, the origin of the interconnection is predefined-time stable in time $t \le t_p$.

Proof: Consider the positive definite storage function V.

(1) Since G_1 is PTSSP, it holds $\dot{V} \le y^\top u - W - \frac{\eta \Psi(V)\left(e^{\Psi(V)}-1\right)}{e^{\Psi(V)}(t_p-t)}$ for $t_0 \le t < t_p$, with $\eta \in \mathbb{R}_{>1}$, W as a positive definite function and class $\mathcal{K}$ function Ψ, and $\dot{V} \le y^\top u - W$ for $t \ge t_p$. Substituting $u = -ky$ by hypothesis (1), then structure of the closed loop system, $\dot{V} \le -y^\top ky - W - \frac{\eta \Psi(V)\left(e^{\Psi(V)}-1\right)}{e^{\Psi(V)}(t_p-t)}$ for $t_0 \le t < t_p$, and $\dot{V} \le -y^\top ky - W$ for $t \ge t_p$. Suppose $\Psi(V) = \sqrt{V}$ where $V \ge 0$. Then, $\dot{V} \le \begin{cases} -\dfrac{\eta\sqrt{V}\left(e^{\sqrt{V}}-1\right)}{e^{\sqrt{V}}(t_p-t)}, & \text{if } t_0 \le t < t_p \\ -W, & \text{otherwise} \end{cases}$. Let $\xi = \sqrt{V}$, then, its time derivative $\dot{\xi} = \frac{\dot{V}}{2\sqrt{2V}} \le \frac{-\eta(e^{\xi}-1)}{2e^{\xi}(t_p-t)}$ as long as $V(x(t)) > 0$, when $V(x(t_0)) > 0$ which is actually the predefined-time dynamics. Thus, from Lemma 1, one can conclude that $x(t) = 0$ is predefined-time stable because in $t \le t_p$, trajectories converge to the origin and after that $x = 0 \Rightarrow W = 0,\ \forall\, t \ge t_p$. Similar proof one can easily extend for $u = -\varphi(y)$.

(2) G_1 is passive, hence, $\dot{V} \le y^\top u$. Substituting the control from the assumption (2), then the closed loop system, $\dot{V} \le -y^\top \phi(y) - y^\top My$ for $t_0 \le t < t_p$ and $\dot{V} \le -y^\top My$ for $t \ge t_p$. Hence, $\dot{V}$ is negative semi-definite for all $t \ge t_0$ and the interconnection is at least Lyapunov stable. Now, note that $\dot{V} = 0$ in time

$t \leq t_p$ if and only if $y = 0$ in time $t \leq t_p$. By zero-state observability, $y(t) = 0 \implies u(t) = 0 \implies x(t) = 0$ in time $t \leq t_p$. Therefore, by the invariance rule, the origin of the interconnected system is predefined-time stable in $t \leq t_p$. After $t \geq t_p$, $\dot{V} \leq -y^\top M y$, which again infers that the origin is asymptotically stable. ■

5. EXAMPLES

Example 1. Consider an example of a time-varying mechanical system, where the mass is decreasing with respect to time: $m(t) = m_q e^{-\alpha t} + m_0$, where m_0 is the mass with no fuel, m_q is the mass of the fuel and α is the decay rate. This system is similar to a space rocket problem where mass of the rocket decreases with time as the fuel is expelled during launch. The dynamical equation of this system is given by: $m(t)\ddot{x}(t) + \frac{1}{2}\dot{m}(t)\dot{x}(t) + kx(t) = u(t)$. This equation can be represented in the state-space form:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{-k}{m(t)}x_1(t) - \frac{1}{2}\frac{\dot{m}(t)}{m(t)}x_2(t) + \frac{u(t)}{m(t)};\ y = x_2(t) \end{aligned} \tag{7}$$

where x_1, x_2 denote the position and velocity, u is the input, y is the output and $k > 0$ is the constant gain. We wish to design u to stabilize at the point $x_1 = x_2 = 0$ in the predefined time set a priori. Now let us first check that the system (7) is passive with output $y = x_2$. Let the storage function be $V = \frac{1}{2}kx_1^2(t) + \frac{1}{2}m(t)x_2^2(t)$. The rate of change of the storage function along the system trajectories (7) is given by $\dot{V} \leq ux_2$, that is, $\dot{V} \leq uy$, which shows that the system is passive. Consider the feedback control: $u(t) := \begin{cases} \dfrac{-\eta(e^y - 1)}{e^y(t_p - t)} - ay, & \text{if } t_0 \leq t < t_p \\ -ay, & t \geq t_p \end{cases}$, where $\eta \in \mathbb{R}_{>1}$ and a is a positive constant. It can be easily observed that the system (7) is zero-state observable, since $x_2 = 0 \implies x_1 = 0$. Applying Theorem 2, x_1, x_2 reach the origin in $t \leq t_p$ with control u as shown in Fig. 2, where t_p is chosen as 4 sec and $m_0 = 1.5, k = 5, \alpha = 0.5, \eta = 2.2, a = 1$. After $t > t_p$, x_1 and x_2 remain at the origin according to the structure of the designed control u.

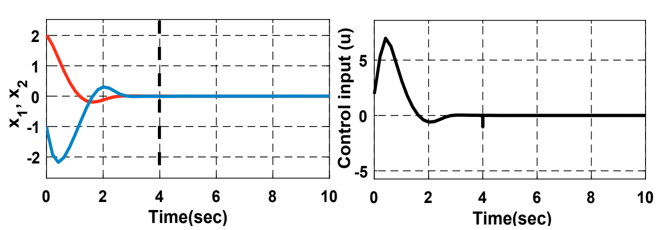

Fig. 2. States evolutions and control input of Example 1 with $t_p = 4$ sec

Example 2. Consider a pendulum system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{ml^2}(-kx_1 - mgl\sin x_1 + u); \quad y = x_2 \end{aligned} \tag{8}$$

where x_1 and x_2 are the angular position and angular velocity respectively, and m, g, l and $k > 0$ are constant system parameters. The input to the system is u and the system output is y. Now, let us first check that the system (8) is passive with output $y = x_2$. Let the storage function be $V(x_1, x_2) = \frac{1}{2}kx_1^2 + \frac{1}{2}ml^2x_2^2 + mgl(1 - \cos x_1)$. The rate of change of the storage function along the system trajectories (8): $\dot{V} \leq ux_2$, that is, $\dot{V} \leq uy$, which shows that the system is passive. Consider the feedback control: $u := \begin{cases} \dfrac{-\eta(e^y - 1)}{e^y(t_p - t)} - ay, & \text{if } t_0 \leq t < t_p \\ -ay, & t \geq t_p \end{cases}$, where $\eta \in \mathbb{R}_{>1}$ and a is a positive constant. It can be easily observed that the system (8) is zero-state observable, since $x_2 = 0 \implies x_1 = 0$. Applying Theorem 2, x_1, x_2 reach the origin in $t \leq t_p$ with control input u as shown in Fig. 3, where t_p is chosen as 5 sec and $m = 1, g = 9.8, l = 0.5, k = 5, \eta = 2.5, a = 2$. After $t > t_p$, x_1 and x_2 remain at the origin according to the structure of the designed control.

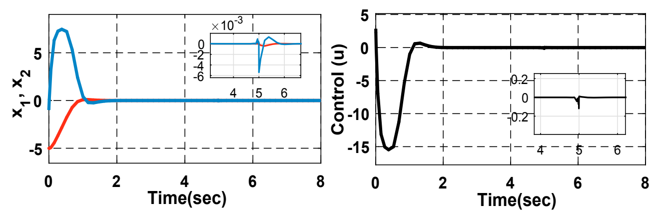

Fig. 3. States evolutions and control input of Example 2 with $t_p = 5$ sec

Example 3. Consider a model of a $m-$link robot (Berghuis and Nijmeijer (1993)),

$$\mathcal{Q}(x)\ddot{x} + \mathcal{C}(x, \dot{x})\dot{x} + h_g(x) = u \tag{9}$$

where $x \in \mathbb{R}^m$ represents the position of the joints, $u \in \mathbb{R}^m$ is a torque, $\mathcal{Q}(x)$ is a symmetric positive definite matrix, $h_g(x)$ represents the forces due to gravity and $\mathcal{C}(x, \dot{x})\dot{x}$ denotes the centrifugal and Coriolis forces. Our objective is to design torque u to stabilize at the point $x = 0, \dot{x} = 0$ in the predefined time t_p, however it is not an equilibrium point when $u = 0$. Let the torque be: $u = h_g(x) - k_1 x + \nu$, where k_1 is a symmetric positive definite matrix and ν is an input to be chosen appropriately. After substituting u, the model becomes

$$\mathcal{Q}(x)\ddot{x} + \mathcal{C}(x, \dot{x})\dot{x} + k_1 x = \nu \tag{10}$$

Let us consider a storage function: $V = \frac{1}{2}\dot{x}^\top \mathcal{Q}\dot{x} + \frac{1}{2}x^\top k_1 x$. The derivative of V along the system (10) is given by

$$\dot{V} = \frac{1}{2}\dot{x}^\top(\dot{\mathcal{Q}}(x) - 2\mathcal{C})\dot{x} - \dot{x}^\top k_1 x + \dot{x}^\top \nu + \dot{x}^\top k_1 \dot{x} \leq \ \dot{x}^\top \nu$$

Let us define the output as $y = \dot{x}$. Then, the system with input ν and output y is passive. In our case, we consider $m = 2$. Further, it can be noticed that with $\nu = 0$, $y = 0 \implies \dot{x} = 0 \implies \ddot{x} = 0 \implies k_1 x = 0 \implies x = 0$, which infers that the system is zero-state observable. Hence, from Theorem 2, system (9) can be predefined-time stabilized in time t_p by the feedback control: $\nu := \begin{cases} -[\phi(y_1), \phi(y_2)]^\top - k_2 y, & \text{if } t_0 \leq t < t_p \\ -k_2 y, & t \geq t_p \end{cases}$, where $y = [y_1, y_2]^\top$, $\phi(y_i) = \frac{\eta_i(e^{y_i} - 1)}{e^{y_i}(t_p - t)}$, $i = 1, 2$, $\eta_i \in \mathbb{R}_{>1}$ and k_2 is a symmetric positive definite matrix. The simulation results are shown in Fig. 4 which shows that the states reach the origin in predefined time $t_p = 5$ sec and $t_p = 7$ sec with control input u, where k_1 and k_2 are the matrices with diagonal entries 50 and 20 and $\eta_1 = \eta_2 = 2.5$. The model (9) parameters values are taken from Berghuis and Nijmeijer (1993).

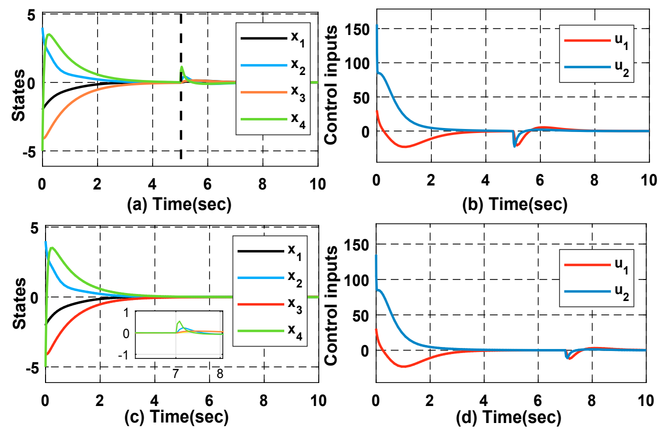

Fig. 4. States evolutions and control inputs of Example 3 with $t_p = 5$ sec (a)(b) and $t_p = 7$ sec (c)(d)

Now, we perform comparison of our proposed control approach with the existing techniques for instance finite-time passivity based control Wang et al. (2017) on the pendulum system described in Example 2. Fig. 5 (a) illustrates the proposed approach with predefined time $t_p = 5$ sec, $\eta = 2.5$, and finite time control implementation $(u := -|y|^{1/2}\text{sign}(y))$ is shown in Fig. 5 (b). Initial conditions $[x_1(0), x_2(0)] = [-17, 20]$ remain the same for both simulations.

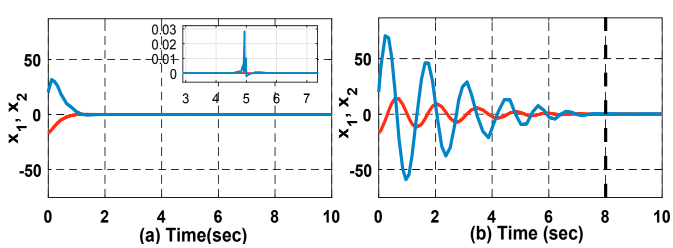

Fig. 5. Comparison between proposed approach (a) and finite-time control (b) with same initial conditions

6. CONCLUSION

A dimension in the framework of passivity, known as predefined-time passivity has been presented. A series of definitions such as PTOSP and PTSSP are discussed to relate the passivity theory with predefined-time stable systems. Further, predefined-time passivity-based control methodology is developed that renders a passive system, predefined-time stable about an equilibrium point by applying a feedback control. Several feedback interconnections of these systems are discussed to investigate predefined-time passivity properties. The effectiveness of this scheme is well illustrated through realistic examples and comparison is also done with other existing techniques.

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