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Graph width parameters: from structure to algorithm

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Graph Width Parameters: from Structure to Algorithms

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Abstract

Solving a discrete optimization problem means seeking an optimal solution from finitely many options. Most discrete optimization problems are computationally hard. To overcome this, we may restrict the input and ask: Which input restrictions lead to "fast" algorithms? The input is often described by a graph and knowing that a graph is "easily decomposable" can be highly useful for designing efficient algorithms for many well-known optimization problems.

A graph width parameter p is a function that assigns a number to each graph G, where a small value of $p(G)$ usually means that the graph G is "easily decomposable" with respect to the parameter p. We use graph width parameters to show that certain graphs are "easily decomposable" with respect to specific graph width parameters and design "fast" algorithms for otherwise "hard" problems.

In Chapter 2 and Chapter 3 we give basic definitions and background for the graph width parameters that are studied in this thesis, which are mim-width (Chapter 4), sim-width (Chapter 5), layered tree-independence number and the related notion of fractional tree-independence-number-fragility (Chapter 7). In Chapter 4 we continue the work from Brettell et al. [27] and prove (un)boundedness of mim-width for (H_1, H_2)-free graphs when H_1 is complete or edgeless, closing most of the open cases. In Chapter 5 we show that, for odd d, List (d,k)-Colouring is polynomial-time solvable parameterized by the sim-width of a given branch decomposition and prove several other properties of this parameter. In Chapter 6 we study the relationships between sim-width, mim-width, tree-independence number, treewidth, clique-width and twin-width when restricted to $K_{t,t}$-free graphs, $K_{t,t}$-subgraph-free graphs and line graphs. In particular, we show that, with the exception of twin-width, all these width parameters are equivalent when restricted to $K_{t,t}$-subgraph-free graphs. In Chapter 7 we study the notion of fractional tree-independence-number-fragility and show that, for every graph in a fractionally fragile graph class, the meta-problem of finding a subset of its vertices satisfying a given CMSO_2 formula and inducing a subgraph of bounded clique size admits a polynomial-time approximation scheme.

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My most heartfelt gratitude goes to my parents whose support and comfort has been invaluable to me my whole life.

I dedicate this thesis to *Adachi and Shimamura*

Contents

Chapter 1

Introduction

A (simple) graph consists of a set of vertices and a set of edges that join pairs of vertices. Graphs allow us to model many computational problems. Route maps, for example, can be considered as graphs, where each location can be viewed as a vertex and the roads connecting them can be viewed as edges. In a map application, we can ask to find the shortest route from one location to another. Loosely speaking, from the perspective of graphs, this is the same as asking for the shortest path between two vertices of a graph, a problem known as Shortest Path.

Consider another example that may appear less graph-like. Say a company is planning to build windmills on a large wind farms. There are some possible (fixed and finitely many) locations to build windmills, but it is not ideal to build two windmills too close to each other since there will be a wind-shadowing effect reducing the power output. If the company wishes to construct as many windmills as possible but does not want to place two windmills within two hundred metres of each other, how can we find an optimal arrangement? We could check all possible arrangements and find the best one by brute force, but this can be computationally infeasible. To find the optimal solution faster, we can restate this problem in graph-theoretical terms, where we treat each possible windmill location as a vertex and add an edge between two vertices if their respective locations are within two hundred metres of each other. Then, an optimal placement of windmills is equivalent to a maximum-size set of vertices in the graph such that no two vertices are joined by an edge. This problem is known as Independent Set. Notice however that in real-world applications there can be other factors to consider. For example, each different location may generate power at different rates and there are costs to construct and maintain each windmill. We can still capture this scenario using graphs. For each vertex, we may assign it a *weight* equal to the expected profit that this windmill will generate. Then, an allocation of windmills that provides maximum profit is equivalent to a set of vertices in the graph with a maximum sum of weights such that no two vertices are joined by an edge. This problem is known as Max Weight Independent Set.

The three problems described above all have well-established algorithms, but with different running times. Shortest Path is "easy", that is, we can find an optimal solution "fast". On the other hand, Independent Set and Max Weight Independent Set are "hard", that is, there are no known algorithms (and most computer scientists believe that there do not exist such algorithms) to always find optimal solutions "fast". In this thesis, we study whether an otherwise "hard" problem becomes "easy" if we restrict the input graphs to some specific classes and, if so, we try to understand the "structural" reasons for the existence of such algorithms. Consider the windmill example, where we treat each possible location of a windmill as a vertex and add an edge if two locations are within two hundred metres of each other. There is an alternative way to construct this graph: For each possible location of windmills, we draw a disk of radius one hundred metres and connect two vertices if the corresponding disks overlap. This gives us exactly the same graph. Graphs that can be obtained in this way are called *disk graphs*, and in case all disks have the same radius, *unit disk graphs*. Some "hard" problems become either "easy" (for example, Max Clique) or "easy to approximate" (for example, Max Weight Independent Set) if the graphs in input are unit disk graphs.

In this thesis, we use graph width parameters, measures of complexity of graphs, to find under which conditions we can obtain "fast" exact or approximation algorithms for otherwise "hard" problems.

Chapter 2

Preliminaries

2.1 Basic Set Theory notions

In this short section, we review a few basic definitions of Set Theory. Most of the terminology we use is standard and can be found in any textbook on related subjects. $\mathbb{N}$ refers to the set of positive integers.

Given a set A,

Power Set The *power set* of A, denoted as 2^A, is the set of all subsets of A.

Set Family A *set family* is a set of sets. For clarity, we will use calligraphic upper-case letters for set families, upper-case letters for sets and lower-case letters for elements when possible.

Partition A set family $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ is called a *partition* of U if $U = P_1 \cup P_2 \cup \ldots \cup P_k$ and for all i, j such that $1 \leq i < j \leq k$ we have $P_i \cap P_j = \varnothing$ and $P_i \neq \varnothing$.

Co-finite A set of natural numbers $A \subseteq \mathbb{N}$ is *co-finite* if $\mathbb{N} \setminus A$ is finite.

2.2 Basic Graph Theory notions

In this section, we give a short overview of some graph terminology that will be relevant. See for example a book by West [151] and a book by Diestel [60].

A (simple, undirected) graph $G = (V(G), E(G))$ is a pair of a vertex set $V(G)$ and an edge set $E(G)$, where each element of $E(G)$ is an unordered pair of distinct elements of $V(G)$. We work with finite graphs and there are no loops or parallel edges unless stated otherwise. For vertices $u, v \in V(G)$, we say u is adjacent to v if $uv \in E(G)$.

Path and Cycles A *path* is a finite sequence of edges which joins a sequence of distinct vertices. If the path starts from vertex u and ends at vertex v, this path is referred to as a *uv-path*. The length of the path equals the number of edges in the path. P_n refers to a path on n vertices. A *cycle* is a finite sequence of edges which joins a sequence of distinct vertices and return to the initial vertex. C_n refers to a cycle on n vertices.

Neighbourhood For a vertex $v \in V$, the *(open) neighbourhood* $N(v)$ is the set of vertices adjacent to v in G. The *closed neighbourhood* of v, $N[v]$, equals $\{v\} \cup N(v)$.

Degree The *degree* $d(v)$ of a vertex $v \in V$ is the size of its neighbourhood, $|N(v)|$.

Subcubic A graph is *subcubic* if every vertex has degree at most 3.

Distance The *distance* from a vertex u to a vertex v in G is the length of a shortest path between u and v.

Bipartite, r-Partite and Co-bipartite A graph is *r-partite*, for $r \geq 2$, if its vertex set admits a partition into r classes such that every edge has its endpoints in different classes. An r-partite graph in which every two vertices from different partition classes are adjacent is a *complete r-partite graph*, and a 2-partite graph is also called *bipartite*. The complete bipartite graph, or a biclique, with partition classes of size t and s is denoted by $K_{t,s}$. A graph is *co-bipartite* if it is the complement of a bipartite graph.

Clique and Independent Set A *clique* of a graph G is a set of pairwise adjacent vertices and the maximum size of a clique of G is denoted by $\omega(G)$. A clique of size t is denoted as K_t. An *independent set* of a graph G is a set of pairwise non-adjacent vertices and the maximum size of an independent set of G is denoted by $\alpha(G)$.

Tree A *tree* is a connected graph that does not contain any cycle as a subgraph.

Null graph A graph is *null* if it has no vertices.

Complete and Anticomplete For disjoint $S, T \subseteq V$, we say that S is *complete to* T if every vertex of S is adjacent to every vertex of T, and S is *anticomplete to* T if there are no edges between S and T.

Caterpillar For $\ell \geq 1$, an *ℓ-caterpillar* is a subcubic tree T on 2ℓ vertices with $V(T) = \{s_1, \ldots, s_\ell, t_1, \ldots, t_\ell\}$, such that $E(T) = \{s_i t_i \ : \ 1 \leq i \leq \ell\} \cup \{s_i s_{i+1} \ : \ 1 \leq i \leq \ell - 1\}$. The vertices $t_1, t_2, \ldots, t_\ell$ are the leaves and the path $s_1 s_2 \cdots s_\ell$ is the *backbone* of the caterpillar.

Matching and Induced Matching A *matching* of a graph is a set of edges with no repeated ends. A matching $F \subseteq E(G)$ of G is *induced* if there is no edge in G between vertices of different edges of F.

Isomorphism Let G and H be two graphs. A bijection $\psi \colon V(G) \to V(H)$ is called an *isomorphism* from G to H if for all $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $\psi(u)\psi(v) \in E(H)$. We say G is *isomorphic* to H if there is an isomorphism from G to H.

Induced Bipartite Graph Given a graph G and two disjoint sets of vertices A and B, we let $G[A, B]$ denote the bipartite subgraph of G with vertex set $A \cup B$ and contains exactly all edges with one end in A and the other end in B.

Graph Classes A *class* of graphs is a set of graphs. For a given graph property $\mathcal{P}$, the class of $\mathcal{P}$-graphs is the set of all graphs satisfying $\mathcal{P}$.

Ramsey Number By Ramsey's theorem [59], for each $a, b \in \mathbb{N}$, there exists a smallest integer n such that every graph on at least n vertices contains either a clique of size a or an independent set of size b as induced subgraph, which we will denote by $R(a, b)$.

2.3 Graph operations and Graph Classes

In this section, we define some of the operations on graphs that will be used in the thesis.

Disjoint Union The *disjoint union* $G + H$ of graphs G and H has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We denote the disjoint union of k copies of G by kG.

Complement The *complement* of G is the graph $\overline{G}$ with vertex set $V(G)$ and $uv \in E(\overline{G})$ if and only if $u \neq v$ and $uv \notin E(G)$.

Subgraph and Induced Subgraph A graph H is a subgraph of G if it can be obtained from G by deleting vertices and edges. For $S \subseteq V(G)$, $G[S]$ refers the *induced subgrapph* of G where $V(G[S]) = S$ and $uv \in E(G[S])$ if and only if $u, v \in S$ and $uv \in E(G)$. If H is an induced subgraph of G, we write $H \subseteq_i G$. For a graph H, a graph G is H*-free* (or H*-subgraph-free*) if G has no induced subgraph (or no subgraph) isomorphic to H. For a set of graphs $\{H_1, \dots, H_k\}$, a graph G is $(H_1, \dots, H_k)$*-free* if G is H_i-free for every $i \in \{1, \dots, k\}$.

Subdivision The k*-subdivision* of an edge uv in a graph replaces uv by k new vertices $w_1, \dots, w_k$ with edges uw_1, w_kv and w_iw_{i+1} for each $i \in \{1, \dots, k-1\}$, i.e. the edge is replaced by a path of length $k + 1$.

Contractions Let G be a graph and $S \subseteq V(G)$ be a subset of its vertices. The *contraction* of S in G is the operation of adding a vertex x_S to G, making it adjacent to $N_G(S)$, and then removing S from G. An edge contraction is the operation of contracting the endpoints of an edge.

Minors, Induced Minors and Minor-Free A graph H is called a *minor* of a graph G if we can obtain a graph isomorphic to H by performing a series of vertex deletions, edge deletions, or edge contractions on G. G is H*-induced-minor free* if H is not an induced minor of G

Line Graphs The *line graph* of a graph G, $L(G)$, is the graph with vertex set $E(G)$ and two vertices of $L(G)$ are adjacent if their respective edges share an end point in G.

Graph Power For $p \in \mathbb{N}$, the p*-th power* of a graph G is the graph G^p with vertex set $V(G^p) = V(G)$ where, for distinct $u, v \in V(G^p)$, $uv \in E(G^p)$ if and only if u and v are at distance at most p in G.

Trimming Let T be a tree and let v be a leaf of T. Let u be a vertex of degree at least 3 having shortest distance in T from v and let P be the v, u-path in T. The operation of *trimming* the leaf v consists in deleting from T the vertex set $V(P) \setminus \{u\}$.

2.4 Special graph classes

Chordal Graphs A graph is *chordal* if every induced cycle in the graph has exactly three vertices.

Planar Graphs A graph is *planar* if it has a drawing on $\mathbb{R}^2$ such that no two of its edges cross each other.

Geometric Intersection Graphs An *object* in $\mathbb{R}^d$ is a compact and path-connected subset of $\mathbb{R}^d$. Given a collection $\mathcal{O}$ of objects in $\mathbb{R}^d$, the *intersection graph* of $\mathcal{O}$ is a graph where we create a vertex for each object $o \in \mathcal{O}$ and let two vertices be adjacent if and only if the corresponding objects intersect. When the objects are disks in $\mathbb{R}^2$, the corresponding intersection graph is referred to as a *disk graph*. If the disks have the same radius, the corresponding intersection graph is referred to as a *unit disk graph*. When the objects are rectangles of the same width in $\mathbb{R}^2$, the corresponding intersection graph is referred to as a *unit-width rectangle graph*.

2.5 Computational Complexity and some graph problems

In this section, we give a brief introduction to the basic notions of Computational Complexity. See for instance [46] for a more thorough introduction.

Asymptotic notation For two functions $f, g\colon \mathbb{N} \to \mathbb{R}_{\geq 0}$, we write $f(n) = O(g(n))$ if and only if there exists a real constant $c > 0$ and an integer n_0 such that for all $n > n_0$, $f(n) \leq c \cdot g(n)$. We write $f(n) = o(g(n))$ if and only if for every real constant $c > 0$ there exists an integer n_0 such that for all $n > n_0$, $f(n) < c \cdot g(n)$.

P and NP P refers to the class of problems that can be solved in polynomial time. That is, there exists an algorithm that can answer the problem in time $o(n^c)$, where n is the input size and $c \in \mathbb{N}$ is some constant. NP refers to the class of problems whose solution can be verified in polynomial time. Given a Boolean formula over a set of variables, the SATISFIABILITY problem asks whether there is an assignment of truth values to its variables so that the formula evaluates to true. SATISFIABILITY is the "canonical" NP-hard problem. Any other problem B is NP-hard if there is a polynomial-time reduction from an NP-hard problem A to B. That is, if there exists a polynomial-time algorithm that, given each instance of A, produces an equivalent instance of B. A problem is NP-complete if it belongs to NP and is NP-hard.

FPT and XP Let $p(G) = k$ be a parameter of a graph G where $|V(G)| = n$. When we measure the runtime of an algorithm as a function of both n and k, we call it a parameterized algorithm. We say that a parameterized algorithm is FPT parameterized by k if there exists a function f and a polynomial function poly such that the worst-case running time of the algorithm is $f(k)\mathsf{poly}(\mathsf{n})$. We say that a parameterized algorithm is XP parameterized by k if there exists a function f such that the worst-case running time of the algorithm is $n^{f(k)}$.

PTAS A PTAS, short for polynomial-time approximation scheme, for a maximization problem is an algorithm which takes an instance I of the problem and a parameter $\varepsilon > 0$ and produces a solution within a factor $1-\varepsilon$ of the optimal in time $n^{O(f(1/\varepsilon))}$, for some function f. A PTAS with running time $f(1/\varepsilon)\cdot n^{O(1)}$ is called an efficient PTAS (EPTAS for short).

Independent Set and Max Weight Independent Set Given a graph G, Independent Set asks to find an independent set of G of maximum size. Given additionally a weight function $w\colon V(G) \to \mathbb{Q}_+$, a generalisation of Independent Set is Max Weight Independent Set, which asks to find an independent set of G such that the sum of the weights of the vertices is maximized. (Max Weight) Independent Set is NP-complete.

k-Colouring and List k-Colouring A (proper vertex) *colouring* of a graph $G = (V, E)$ is a mapping $c\colon V \to \{1, 2, \ldots\}$ that gives each vertex $u \in V$ a *colour* $c(u)$ in such a way that, for every two adjacent vertices u and v, we have that $c(u) \neq c(v)$. If for every $u \in V$ we have $c(u) \in \{1, \ldots, k\}$, then we say that c is a *k-colouring* of G. The Colouring is to decide whether a given graph G has a k-colouring for some given integer $k \geq 1$. If k is *fixed*, that is, not part of the input, we call this the k-Colouring problem. A generalisation of k-Colouring is the following: For an integer $k \geq 1$, a *k-list assignment* of a graph $G = (V, E)$ is a function L that assigns each vertex $u \in V$ a *list* $L(u) \subseteq \{1, 2, \ldots, k\}$ of *admissible* colours for u. A colouring c of G *respects* L if $c(u) \in L(u)$ for every $u \in V$. For a fixed integer $k \geq 1$, the List k-Colouring problem is to decide whether a given graph G

with a k-list assignment L admits a colouring that respects L. By setting $L(u) = \{1, \dots, k\}$ for every $u \in V$, we obtain the k-Colouring problem. Both the List k-Colouring problem and the k-Colouring problem are NP-complete.

Chapter 3

Graph Width Parameters and Summary of Main Results

As we discussed in the introduction, there are many natural computational problems that are NP-hard, so we do not expect fast (polynomial-time) algorithms for them. In this case, we may restrict the input to some specific graph class with special structure and hope that this structure can help us construct a polynomial-time algorithm. For instance, Max Weight Independent Set is NP-hard in general but admits polynomial-time algorithms on chordal graphs. Therefore, it is interesting to investigate whether we can find polynomial-time algorithms for otherwise NP-hard problems when we restrict the input to some graph class $\mathcal{G}$.

Over the last decades, graph width parameters have proven to be an extremely successful tool in algorithmic graph theory. We start our study with mim-width, short for maximum induced matching width, which is defined in terms of branch decompositions.

Branch Decomposition Given a set S, a function $f\colon 2^S \to \mathbb{Z}$ is *symmetric* if $f(X) = f(\overline{X})$ for all $X \subseteq S$, where we use $\overline{X}$ to denote $S \setminus X$. A *branch decomposition on* S is a pair (T, δ), where T is a subcubic tree and δ is a bijection between S and the leaves of T. Each edge $e \in E(T)$ naturally splits the leaves of the tree in two parts depending on what component they belong to when e is removed. In this way, each edge $e \in E(T)$ represents a partition of S into two partition classes that we denote A_e and $\overline{A_e}$. Let $f\colon 2^S \to \mathbb{Z}$ be a symmetric function and let (T, δ) be a branch decomposition on S. The *f-width of* (T, δ) is the quantity $\max_{e \in E(T)} f(A_e)$. The *f-branch-width on S* is either the minimum f-width over all branch decompositions on S when $|S| \geq 2$ or $f(\varnothing)$ when $|S| \leq 1$.

Mim-width For a branch decomposition of G with $S = V(G)$ and $X \subseteq V(G)$, let $\mathrm{cutmim}_G(X)$ be the size of a maximum induced matching in $G[X, \overline{X}]$. The *mim-width of* G, denoted $\mathrm{mimw}(G)$, is the cutmim_G-branch-width on V.

In general, computing the mim-width is NP-hard and there is no polynomial-time algorithm for approximating the mim-width of a graph to within a constant factor of the optimal unless NP = ZPP [141]. Moreover, it remains a challenging open problem to obtain a polynomial-time algorithm for computing a branch decomposition with mim-width $f(k)$ of a graph with mim-width k. Therefore, in contrast to algorithms for graph classes of bounded treewidth or rank-width [17, 135], algorithms for classes of bounded mim-width require a branch decomposition of constant mim-width as part of the input. Obtaining such branch decompositions in polynomial time has been shown to be possible for several special graph classes $\mathcal{G}$ (see, e.g., [10, 26]). In this case, we say that the mim-width of $\mathcal{G}$ is *quickly computable*.

A&C DN logic is a logic framework introduced by Bergougnoux et al. [13], who showed that any problem expressible in A&C DN logic can be solved in XP time parameterized by the mim-width of a given branch decomposition of the input graph. Examples of such problems include Max Weight Independent Set and k-Colouring. Therefore, if we are interested in showing that a problem expressible in A&C DN logic is polynomial-time solvable when restricted to a specific graph class, instead of designing an ad-hoc algorithm for this problem, we can simply try to show that the graph class has bounded mim-width and that a branch decomposition witnessing this can be computed in polynomial time. This approach is often easier and can prove multiple results at once.

In Chapter 4 we make progress towards classifying the mim-width of (H_1, H_2)-free graphs in the case H_1 is complete or edgeless. Brettell et al. [26] determined the mim-width (un)boundedness of (H_1, H_2)-free graph classes for most pairs of H_1, H_2. They considered walls, a class of graphs of unbounded mim-width, and various operations on graphs and determined their effect on mim-width. In particular, they showed that adding edges between vertices in the same vertex class of a k-partite graph decreases the mim-width by at most $(k-1)/k$ and that edge subdivisions do not decrease the mim-width of a graph. Using these results in a novel way, we are able to prove that several graph classes have unbounded mim-width, thus solving some open problems from [26].

(1.A) For each $t \geq 4$, the mim-width of $(3P_1, \overline{K_{3,t} + P_1})$)-free graphs and the mim-width of $(4P_1, \overline{K_{2,t} + P_1})$-free graphs are bounded and quickly computable.

(1.B) The class of $(3P_1, \overline{K_{4,4} + P_1})$-free graphs, the class of $(4P_1, \overline{K_{3,3} + P_1})$-free graphs and the class of $(5P_1, \overline{K_{2,2} + P_1})$-free graphs have unbounded mim-width.

(1.C) The class of $(K_5, P_3+P_2+P_1)$-free graphs and the class of $(K_4, P_3+2P_2+P_1, 2P_3+P_2)$-free graphs have unbounded mim-width.

In Chapter 5 we study sim-width, short for special induced matching width. There are many important graph classes with unbounded mim-width but still admitting polynomial-time algorithms for several NP-hard problems. Consider, for example, the class of co-comparability graphs, i.e., intersection graphs of curves between two parallel lines. It is known that 3-Colouring and Independent Set are polynomial-time solvable when restricting to this class [143, 31]. However, co-comparability graphs have unbounded width with respect to most well-studied graph width parameters, so it would be desirable to construct a graph width parameter that can explain this jump in computational complexity. For this purpose, Kang et al. [111] defined sim-width.

Sim-width For a branch decompostion of G with $S = V(G)$ and $X \subseteq V$, let $\text{cutsim}_G(X)$ be the size of a maximum induced matching between A_e and $\overline{A_e}$ in G. The *sim-width of* G, denoted $\text{simw}(G)$, is the cutsim_G-branch-width on V.

Remark 3.1. That is, when measuring mim-width, we "ignore" edges between vertices from the same partition class when finding the maximum induced matching. For sim-width, we do not ignore these edges. That is, the vertices from each partition class must induce an independent set. Similar to mim-width, there is no known polynomial-time algorithm for computing a branch decomposition with sim-width $f(k)$ of a graph with sim-width k [105])

We begin our study of sim-width with a structural result: we show that boundedness of sim-width of graph powers is preserved precisely in the case of odd powers.

(2.A) The sim-width of an odd power of a graph is at most the sim-width of the original graph.

Using result (2.A), we prove the following algorithmic result:

(2.B) For odd d, List (d,k)-Colouring is polynomial-time solvable for every graph class whose sim-width is bounded and quickly computable.

We also consider Max Independent Packing and show that

(2.C) If Independent Set is polynomial-time solvable for graph classes of bounded sim-width, then so is Max Independent Packing.

In Section 5.5, we prove the following result which has an important consequence in Chapter 6 (result 3.D):

(2.D) The sim-width of a line graph $L(G)$ does not increase by performing edge contractions on the original graph G.

It is in general non-trivial to determine the exact values of $p(L(K_{n,m}))$ and $p(L(K_n))$, for some width parameter p. For $n \geq 3$, Lucena [125] showed that $\text{tw}(L(K_{n,n})) = n^2/2 + n/2 - 1$ (where $\text{tw}(G)$ denotes the treewidth of the graph G). This result was extended by Harvey and Wood [99], who showed that $\text{tw}(L(K_{n,m}))$ has order nm. Moreover, Harvey and Wood [98] determined the exact value of $\text{tw}(L(K_n))$. We conclude Chapter 5 by showing that:

(2.E) For $n \geq 6$, $\text{simw}(L(K_{n,m})) = \lceil n/3 \rceil$.

It is then natural to compare different width parameters. To do this, we introduce the following notion:

Domination Between Graph Width Parameters We say that a width parameter p *dominates* a width parameter q if there exists a function f such that $p(G) \leq f(q(G))$ for every graph G. If p dominates q but q does not dominate p, then p is *more powerful* than q. If p

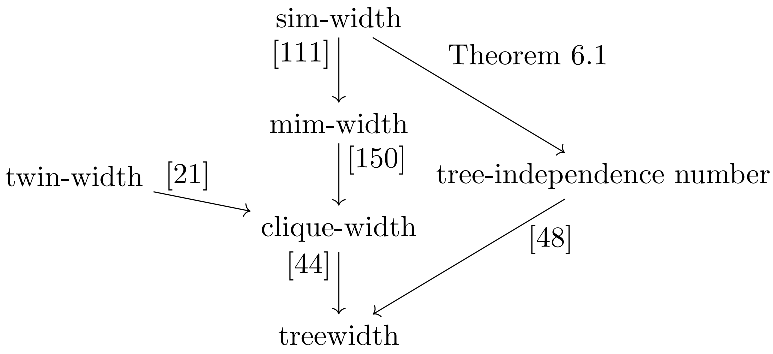

Figure 3.1: The relationships between the different width parameters that we consider. Parameter p is more powerful than parameter q if and only if there exists a directed path from p to q. To explain the incomparabilities: complete bipartite graphs have clique-width 2 but unbounded tree-independence number [48], chordal graphs have tree-independence number 1 [48] but unbounded twin-width [22] and mim-width [111]. Walls have bounded twin-width [21] but unbounded sim-width (Theorem 6.8).

dominates q and q dominates p, then p and q are *equivalent*. In particular, if two equivalent parameters p and q admit linear functions witnessing this, we say that p and q are *linearly equivalent* (in other words, one is a constant factor approximation of the other). If neither p dominates q nor q dominates p, then p and q are *incomparable*. A width parameter p is *bounded* on a graph class $\mathcal{G}$ if there exists a constant c such that $p(G) \leq c$ for every $G \in \mathcal{G}$. Note that if p and q are two equivalent width parameters then, for every graph class $\mathcal{G}$, the parameter p is bounded on $\mathcal{G}$ if and only if q is bounded on $\mathcal{G}$.

In Chapter 6 we study how the relationship between non-equivalent width parameters changes once we restrict to some special graph class. As width parameters, we consider treewidth, clique-width, twin-width, mim-width, sim-width and tree-independence number, whereas as graph classes we consider $K_{t,t}$-subgraph-free graphs, line graphs, and their common superclass, for $t \geq 3$, of $K_{t,t}$-free graphs.

The study of equivalence of width parameters arguably starts with two well-known results of Gurski and Wanke characterizing clique-width on $K_{t,t}$-subgraph-free graphs [95] and line graphs [96] in terms of treewidth. The first result [95] asserts that clique-width and treewidth are equivalent for the class of $K_{t,t}$-subgraph-free graphs, for every $t \geq 2$. This implies that rank-width and treewidth are also equivalent for this graph class, and an improved upper bound for treewidth which is polynomial in rank-width was obtained by Fomin, Oum and Thilikos [82].

The second result of Gurski and Wanke [96] asserts that, for any graph class $\mathcal{G}$, the class of line graphs of graphs in $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded treewidth. In particular, although clique-width and treewidth are equivalent for the class of $K_{t,t}$-subgraph-free graphs (one of the least restrictive classes of sparse graphs), clique-width is more powerful than treewidth for line graphs (one of the most prominent classes of dense graphs).

We now recall the definitions of treewidth and tree-independence number, the main graph width parameters we are going to work with, and defer the definitions of the other parameters to Chapter 6.

Tree decomposition A *tree decomposition* of a graph G is a pair $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$, where T is a tree whose every node t is assigned a vertex subset $X_t \subseteq V(G)$, called a *bag*, such that the following conditions are satisfied:

(T1) Every vertex of G belongs to at least one bag;

(T2) For every $uv \in E(G)$, there exists a bag containing both u and v;

(T3) For every $u \in V(G)$, the subgraph T_u of T induced by $\{t \in V(T) : u \in X_t\}$ is connected.

Treewidth Given a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$, its *width* is the maximum value of $|X_t| - 1$ over all $t \in V(T)$. The *treewidth* of a graph G, denoted $\text{tw}(G)$, is the minimum width of a tree decomposition of G.

Tree-Independence Number The *independence number* of $\mathcal{T}$, denoted $\alpha(\mathcal{T})$, is the quantity $\max_{t \in V(T)} \alpha(G[X_t])$. The *tree-independence number* of a graph G, denoted tree-$\alpha(G)$, is the minimum independence number of a tree decomposition of G.

Our main technical contribution is the following:

(3.A) For $K_{t,t}$-free graphs, bounded mim-width implies bounded tree-independence number.

Result (3.A) has several interesting consequences. For example, it implies a special case of a conjecture of Dallard et al. [51]. It also helps us to provide a complete comparison of the mentioned parameters when restricted to $K_{t,t}$-subgraph-free graphs, thus extending the well-known result of Gurski and Wanke [95] stating that treewidth and clique-width are equivalent for the class of $K_{t,t}$-subgraph-free graphs:

(3.B) For $K_{n,n}$-subgraph-free graphs, treewidth, clique-width, mim-width, sim-width and tree-independence number are all equivalent.

Finally, in Chapter 6, using result (2.D), we provide a complete comparison of the mentioned parameters when restricted to line graphs. Extending a result of Gurski and Wanke [96] stating that a class of graphs $\mathcal{G}$ has bounded treewidth if and only if the class of line graphs of graphs in $\mathcal{G}$ has bounded clique-width, we show that:

(3.C) For any class of graphs $\mathcal{G}$ and the class of its line graph $\mathcal{H}$, the clique-width, mim-width, sim-width and tree-independence number are all equivalent and bounded on $\mathcal{H}$ if and only if $\mathcal{G}$ has bounded treewidth.

All width parameters mentioned above are used to design exact polynomial-time algorithms. However, it often happens that a computationally interesting problem remains NP-hard even after restricting to a specific graph class. Recall our windmill example in the introduction. The intersection graphs for the windmills are unit disk graphs, on which Independent Set remains NP-hard. Hence, we should not expect any exact polynomial-time algorithm for this problem. It is then natural to turn to the realm of approximation algorithms and investigate whether there exists a polynomial-time approximation scheme (PTAS) for the problem we are considering. In Chapter 7, we consider the following notion, which turns out to be extremely useful in the design of PTASes for NP-hard graph problems.

Fractional p-Fragility For $\beta \leq 1$, a β-general cover of a graph G is a multiset $\mathcal{C}$ of subsets of $V(G)$ such that each vertex belongs to at least $\beta|\mathcal{C}|$ elements of the cover. For a parameter p, the p-width of the cover is $\max_{C \in \mathcal{C}} p(G[C])$. A graph class $\mathcal{G}$ is fractionally p-fragile if there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for every $r \in \mathbb{N}$, every $G \in \mathcal{G}$ has a $(1 - 1/r)$-general cover with p-width at most $f(r)$. A fractionally p-fragile class $\mathcal{G}$ is efficiently fractionally p-fragile if there exists an algorithm that, for every $r \in \mathbb{N}$ and

$G \in \mathcal{G}$, returns in $\mathsf{poly}(|V(G)|)$ time a $(1-1/r)$-general cover $\mathcal{C}$ of G and, for each $C \in \mathcal{C}$, a tree decomposition of $G[C]$ of width (if $p = \text{tw}$) or independence number (if $p = \text{tree-}\alpha$) at most $f(r)$, for some function $f\colon \mathbb{N} \to \mathbb{N}$.

Dvořák [67] introduced the notion of fractional tw-fragility and showed that it can be used to provide PTASes for some graph problems. Several interesting graph classes are fractionally tw-fragile, for example planar graphs and more generally proper minor-closed graph classes. However, this notion falls short of capturing geometric intersection graphs. To overcome this, we investigate a relaxation of the notion of fractional tw-fragility, namely fractional tree-α-fragility. In particular, we obtain polynomial-time approximation schemes for meta-problems such as finding a maximum-weight sparse induced subgraph satisfying a given CMSO_2 formula on fractionally tree-α-fragile graph classes:

(4.A) For each fixed $c, h \in \mathbb{N}$ and CMSO_2 formula ψ, (c, h, ψ)-Max Weight Induced Subgraph admits a PTAS on every efficiently fractionally tree-α-fragile class.

(4.B) Max Weight Independent $\mathcal{H}$-Packing admits a PTAS on every efficiently fractionally tree-α-fragile class.

(4.C) Every class of intersection graphs of c-fat objects in $\mathbb{R}^d$, for fixed d, is efficiently fractionally tree-α-fragile.

(4.D) For each fixed even $p \in \mathbb{N}$, Max Weight Distance-p $\mathcal{H}$-Packing admits a PTAS on every class of bounded layered tree-independence number[1] and on every class of intersection graphs of c-fat objects in $\mathbb{R}^d$, for fixed d.

The exact definitions of some of the notions above are rather involved, so we leave them to the relevant chapters. However, just to give some examples, collections of c-fat objects include collections of disks and rectangles with bounded aspect ratio but not arbitrarily thin rectangles. The meta-problem (c, h, ψ)-Max Weight Induced Subgraph includes, for example, Max Weight Independent Set and Max Weight Induced Matching.

[1] Provided that a tree decomposition and a layering witnessing small layered tree-independence number can be computed efficiently.

Chapter 4

Mim-Width

This chapter contains joint work with Andrea Munaro: *On algorithmic applications of sim-width and mim-width of* (H_1, H_2)*-free graphs* [130].

4.1 Introduction

Mim-width was first defined by Vatshelle [150] in 2012. They used dynamic programming to show that (σ, ρ)-vertex subset problems are polynomial-time solvable when the input is restricted to be graphs of bounded mim-width with respect to given branch decompositions [32].

Definition 4.1. Let σ and ρ be finite or co-finite sets of natural numbers. A subset S of vertices is a (σ, ρ)-set of G if

$$\forall v \in V(G) : |N(v) \cap S| \in \begin{cases} \sigma, & \text{if } v \in S, \\ \rho, & \text{if } v \in V(G) \backslash S \end{cases} \tag{4.1}$$

Definition 4.2. Define $d(\mathbb{N} \cup \{0\}) = 0$. For every finite or co-finite set $\mu \subseteq \mathbb{N} \cup \{0\}$, let $d(\mu) = 1 + \min(\max_{x \in \mathbb{N} \cup \{0\}}\{x : x \in \mu\}, \max_{x \in \mathbb{N} \cup \{0\}}\{x : x \notin \mu\})$. Let $d(\sigma, \rho) = \max(d(\sigma), d(\rho))$. Note that since μ is finite or co-finite, $d(\mu)$ is always well defined.

Theorem 4.3 (Bui-Xuan et al. [32]). *For every n-vertex graph G given along with a decomposition tree (T, δ), with $\mathrm{mimw}_G(T, \delta) \leq k$ and $d = d(\sigma, \rho)$. Any (σ, ρ)-vertex subset problem on G can be solved in time $O(n^4 2^{3dk^2})$.*

Many classic vertex set properties can be expressed using (σ, ρ)-sets. For example Independent Set is a $(\varnothing, \mathbb{N} \cup \{0\})$-problem with $d = 1$, Minimum Dominating Set is a $(\mathbb{N} \cup \{0\}, \mathbb{N})$-set problem with $d = 1$.

A variant of (σ, ρ)-vertex subset problem is locally checkable vertex partition problems. In the same paper, it was shown that these problems are polynomial-time solvable when the input is restricted to be graphs of bounded mim-width with given branch decompositions.

Definition 4.4. A degree constraint matrix D_q is a q-by-q matrix with entries being finite or co-finite subsets of natural numbers. A D_q-partition in a graph G is a partition $\{V_1, V_2, \ldots, V_q\}$ of $V(G)$ such that for $1 \leq i, j \leq q$ we have $\forall v \in V_i : |N(v) \cap V_j| \in D_q[i, j]$.

The locally checkable vertex partitioning problems consist of deciding if G has a D_q partition. Many important NP-hard problems, for example, k-Colouring, belongs to the framework of locally checkable vertex partitioning problems, where the diagonal entries of D_q equal $\{0\}$ and the remaining entries equal $\mathbb{N}$.

Theorem 4.5 (Bui-Xuan et al. [32]). *For every graph G given along with a decomposition tree (T, δ) such that $\mathrm{mimw}_G(T, \delta) \leq k$ and $d = \max_{i,j} d(D_q[i, j])$, deciding if G has a D_q-partition, can be done in time $O(n^4 q 2^{3qdk^2})$.*

Jaffke et al. [107] initiated a series of papers studying mim-width, finding several NP-hard problems not in the framework of locally checkable vertex set and vertex partition problem that are polynomial-time solvable when the input is restricted to graphs of bounded mim-width equipped with the respective branch decomposition, for example Longest Induced Path. They also showed that the mim-width of graphs is at most doubled when taking graph powers, so $\mathrm{mimw}(G^k) \leq 2\mathrm{mimw}(G)$, hence showing that the distance version of locally checkable vertex set and vertex partition problems are also polynomial-time solvable when the input is restricted to graphs of bounded mim-width equipped with the respective branch decomposition.

In 2022, a recent and remarkable meta-theorem was provided by Bergougnoux et al. [13]. They showed that all problems expressible in A&C DN logic, an extension of existential MSO_1 logic, can be solved in XP time parameterized by the mim-width of a given branch decomposition of the input graph. This result, which can be viewed as the mim-width analogue of the famous meta-theorems for treewidth [42] and clique-width [41], generalises essentially all the previously known XP algorithms parameterized by mim-width, as A&C DN logic captures both local and non-local problems. Just to name a few problems falling into this framework, we have all Locally Checkable Vertex Subset and Vertex Partitioning problems [10, 32], their distance versions [106] and their connectivity and acyclicity versions [12], Longest Induced Path and Induced Disjoint Paths [107], Feedback Vertex Set [108] and Semitotal Dominating Set [86].

On the other hand, Mengel [129] proved several important properties of mim-width. In particular, he showed that strongly chordal split graphs, co-comparability graphs and circle graphs have unbounded mim-width. Together with some other results, mim-width (un)boundedness was determined for most of the important graph classes in [26]. They also showed that adding edges to vertices in the same vertex class of a bipartite graph decreases the mim-width of the graph by at most a half. Following his idea, more general results were obtained in [26] which were used to show several hereditary graph classes have unbounded mim-width. In view of the discussion above, if we are interested in the computational complexity of a certain graph problem restricted to a special graph class, it is useful to know whether the mim-width of the class is bounded and quickly computable. A systematic study on the boundedness of mim-width for hereditary graph classes, comparable to similar studies on the boundedness of clique-width (see, e.g., [47]) and treewidth [124], was recently intiated in [26]. We say a graph class is *hereditary* if it is closed under vertex deletion. It is well known that hereditary graph classes are exactly those classes characterised by a (unique) set $\mathcal{F}$ of minimal forbidden induced subgraphs. If $|\mathcal{F}| = 1$ or $|\mathcal{F}| = 2$, we say that the hereditary graph class is *monogenic* or *bigenic*, respectively. In [26], boundedness or unboundedness of mim-width has been determined for all monogenic classes and a large number of bigenic classes.

Consider first the class of (rP_1, H)-free graphs. If the mim-width of such a class is bounded and quickly computable, we obtain, for many problems, XP algorithms parameterized by $\alpha(G)$ for the class of H-free graphs. For $r \geq 5$, Brettell et al. [27] completely classified the mim-width of the class of (rP_1, H)-free graphs, except for one infinite family, and asked the following:

Question 1 (Brettell et al. [27]). *For each $r \geq 4$, and for each $s, t \geq 2$, determine the (un)boundedness of mim-width of $(rP_1, \overline{K_{s,t} + P_1})$-free graphs.*

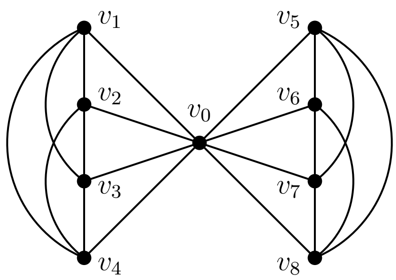

Figure 4.1: The graph $\overline{K_{4,4} + P_1}$.

In Section 4.2, we completely resolve Question 1 and when the mim-width is bounded, our proof also gives a polynomial algorithm for constructing a witnessing branch decomposition.

Theorem 4.6. *Let $r \geq 3$ and $s, t \geq 2$ be integers. Then the mim-width of the class of $(rP_1, \overline{K_{s,t} + P_1})$-free graphs is bounded if and only if:*

- *$r = 3$ and one of s and t is at most 3;*
- *$r = 4$ and one of s and t is at most 2.*

In all these cases, the mim-width is also quickly computable.

Colouring (and hence List Colouring) is NP-complete for circular-arc graphs [89], a class of graphs of mim-width at most 2 and for which mim-width is quickly computable [10]. On the other hand, mim-width has proven to be particularly effective in tackling k-Colouring and List k-Colouring. For instance, Kwon [116] showed the following (see also [27]):

Theorem 4.7 (Kwon [116]). *For every $k \geq 1$, List k-Colouring is polynomial-time solvable for every graph class whose mim-width is bounded and quickly computable.*

Recall that $K_{r,s}$ is the complete bipartite graph with partition classes of size r and s. The 1-subdivision of $K_{1,s}$ is denoted by $K^1_{1,s}$; in particular $K^1_{1,2} = P_5$. Brettell et al. [27] showed that a number of known polynomial-time results for k-Colouring and List k-Colouring on hereditary classes [39, 45, 91, 102] can be obtained, and strengthened, by combining Theorem 4.7 with the following:

Theorem 4.8 (Brettell et al. [27]). *For every $r \geq 1$, $s \geq 1$ and $t \geq 1$, the mim-width of the class of $(K_r, K^1_{1,s}, P_t)$-free graphs is bounded and quickly computable.*

The trivial but useful observation is that each yes-instance of List k-Colouring is K_{k+1}-free, and so we obtain that, for every $k \geq 1$, $s \geq 1$ and $t \geq 1$, List k-Colouring is polynomial-time solvable for $(K^1_{1,s}, P_t)$-free graphs [27]. Hence, in the context of colouring problems on hereditary classes, it makes sense to investigate the mim-width of subclasses of K_r-free graphs. A first step is to consider the mim-width of (K_r, H)-free graphs, for some graph H. For any H such that the mim-width of (K_r, H)-free graphs is bounded and quickly computable, List k-Colouring is polynomial-time solvable for all $k < r$. More generally, for problems admitting polynomial-time algorithms when mim-width is bounded and quickly computable, we obtain XP algorithms parameterized by $\omega(G)$ when restricted to H-free graphs. For example, Chudnovsky et al. [40] showed that for P_5-free graphs, there exists an $n^{O(\omega(G))}$-time algorithm for Max Partial H-Colouring (a common generalisation of Independent Set and Odd Cycle Transversal which is polynomial-time solvable when mim-width is bounded and quickly computable). Theorem 4.8 allows to generalise this, although with a worse running time (see [27, 40]).

From a merely structural point of view, the study of the mim-width of (K_r, H)-free graphs falls into the systematic study of the mim-width of bigenic classes mentioned above. For each $r \geq 4$, Brettell et al. [27] completely classified the mim-width of the class of (K_r, H)-free graphs, except for one infinite family, and asked the following:

Question 2 (Brettell et al. [27]). *For each $r \geq 4$, and for each $t \geq 0$ and $u \geq 1$ such that $t + u \geq 2$, determine the (un)boundedness of mim-width of $(K_r, tP_2 + uP_3)$-free graphs.*

We answer most parts of this open problem:

Theorem 4.9. *Let $r \geq 5$ be an integer and let $H = sP_1 + tP_2 + uP_3$, for $s, t, u \geq 0$. Then exactly one of the following holds:*

- *$H \subseteq_i sP_1 + tP_2$, or $H \subseteq_i sP_1 + P_3$, and the mim-width of the class of (K_r, H)-free graphs is bounded and quickly computable;*
- *$H \supseteq_i P_3 + P_2 + P_1$, and the mim-width of the class of (K_r, H)-free graphs is unbounded;*

- $H = 2P_3$, *or* $H = P_3 + P_2$.

Theorem 4.10. *Let* $r = 4$ *and let* $H = sP_1 + tP_2 + uP_3$, *for* $s, t, u \geq 0$. *Then exactly one of the following holds:*

- $H \subseteq_i sP_1 + tP_2$, *or* $H \subseteq_i sP_1 + P_3$, *and the mim-width of the class of* (K_r, H)*-free graphs is bounded and quickly computable;*
- $H \supseteq_i P_3 + 2P_2 + P_1$, *or* $2P_3 + P_2$, *and the mim-width of the class of* (K_r, H)*-free graphs is unbounded;*
- $H = P_3 + 2P_2$, *or* $H = P_3 + P_2 + sP_1$, *or* $H = 2P_3 + sP_1$.

Our results are related to the class of uP_3-free graphs. Recently, Hajebi et al. [97] showed that, for every $u \geq 1$, List 5-Colouring is polynomial-time solvable for uP_3-free graphs. Since an instance of List 5-Colouring can always be assumed to be K_6-free, in view of Theorem 5.9 an alternative approach to obtaining the aforementioned result might pass through studying the mim-width of (K_6, uP_3)-free graphs. Unfortunately, Theorem 4.9 readily shows that, with the possible exception of the case $u = 2$, this is not possible: For each $u \geq 3$, the mim-width of (K_6, uP_3)-free graphs is unbounded and, by [111, Proposition 4.2], the same must be true for sim-width.

4.2 Mim-width of $(rP_1, \overline{K_{s,t} + P_1})$-free graphs

In this section we show the mim-width dichotomy for the class of $(rP_1, \overline{K_{s,t} + P_1})$-free graphs stated in Theorem 4.6. We begin by identifying the cases of bounded mim-width (Section 4.2.1) and then pass to the cases of unbounded mim-width (Section 4.2.2). These results are then combined to prove Theorem 4.6 (Section 4.4).

4.2.1 Boundedness results

In this section we show that, for each $t \geq 4$, the mim-width of $(3P_1, \overline{K_{3,t} + P_1})$-free graphs and the mim-width of $(4P_1, \overline{K_{2,t} + P_1})$-free graphs are bounded and quickly computable (Theorems 4.14

and 4.17, respectively). The proofs are based on the following common strategy. We find s pairwise non-adjacent vertices $v_1, \ldots v_s$ in the input graph G ($s = 2$ in Theorem 4.14 and $s = 3$ in Theorem 4.17). We then obtain a partition of $V(G)$ where one partition class is $\{v_1, \ldots, v_t\}$ and the remaining ones are the sets of private neighbours of subsets of $\{v_1, \ldots, v_t\}$ with respect to $\{v_1, \ldots, v_t\}$. We finally construct an appropriate branch decomposition of G and use the following simple observation.

Observation 4.11. *Let $V_1, \ldots, V_m$ be a partition of $V(G)$ and let (T, δ) be a branch decomposition of G. Then,*

$$\mathrm{mimw}_G(T, \delta) = \max_{e \in E(T)} \mathrm{cutmim}_G(A_e, \overline{A_e}) \leq \max_{e \in E(T)} \sum_{1 \leq i,j \leq m} \mathrm{cutmim}_G(A_e \cap V_i, \overline{A_e} \cap V_j).$$

We will need two auxiliary results. The first one below is left as an easy exercise (see Figure 4.2).

Lemma 4.12. *Let G be a graph and let (T, δ) be a branch decomposition of G with $\mathrm{mimw}_G(T, \delta) \leq k$. Let G' be the graph obtained from G by adding a vertex of degree at most 1. Then we can construct in $O(1)$ time a branch decomposition (T', δ') of G' with $\mathrm{mimw}_{G'}(T', \delta') \leq k$.*

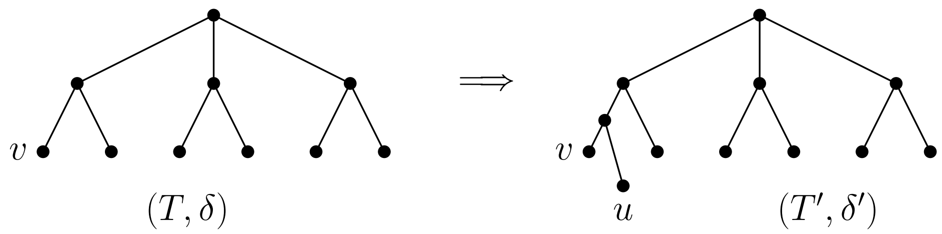

Figure 4.2: How to construct a branch decomposition (T', δ') of G' from a branch decomposition (T, δ) of G, where G' is obtained from G by adding a leaf vertex u adjacent to v.

The second one is essentially stated in the proof of [150, Corollary 3.7.4]. We provide its short proof for completeness.

Lemma 4.13 (Vatshelle [150])**.** *Let G be a graph with $|V(G)| > 1$ and maximum degree at most 2. Then $\mathrm{mimw}(G) \leq 2$ and a branch decomposition (T, δ) of G with $\mathrm{mimw}_G(T, \delta) \leq 2$ can be constructed in $O(n)$ time.*

Proof. Suppose that G has k components, $C_1, \ldots, C_k$, where each C_i is a path or a cycle with vertex set $\{v_{i,1}, \ldots, v_{i,|C_i|}\}$. For $1 < j < |C_i|$, each $v_{i,j}$ is adjacent to $v_{i,j-1}$ and $v_{i,j+1}$ and, if

C_i is a cycle, $v_{i,1}$ is adjacent to $v_{1,|C_i|}$. For each component C_i, we construct a $|C_i|$-caterpillar T_i with leaves $\ell_{i,1}, \dots, \ell_{i,|C_i|}$ and subdivide an arbitrary edge of the backbone of T_i with a new vertex t_i, unless the backbone of T_i has size 1, in which case we let t_i be the unique vertex of the backbone. We then construct a k-caterpillar T_0 with leaves $\ell_{0,1}, \dots, \ell_{0,k}$. Let T be the subcubic tree obtained from the disjoint union of $T_0, T_1, \dots, T_k$ by adding the edges $\ell_{0,1}t_1, \dots, \ell_{0,k}t_k$ and, if $k = 1$, by additionally deleting $V(T_0)$. Let δ be the bijection from the vertices of G to the leaves of T given by $\delta(v_{i,j}) = \ell_{i,j}$. Clearly, (T, δ) is a branch decomposition of G and it can be constructed in $O(n)$ time.

We now show that $\text{mimw}_G(T, \delta) \leq 2$. Let $e \in E(T)$ and consider the partition $(A_e, \overline{A_e})$ of $V(G)$ induced by e. Suppose first that e belongs to $E(T_0)$ or $e = \ell_{0,j}t_j$ for some j. Then, for each component C_i of G, $V(C_i)$ is fully contained in either A_e or $\overline{A_e}$ and so $\text{cutmim}_G(A_e, \overline{A_e}) = 0$. Suppose now that e belongs to the backbone of T_i, for some $i > 0$. Then, it is easy to see that there are are at most two edges across the cut $(A_e, \overline{A_e})$, from which $\text{cutmim}_G(A_e, \overline{A_e}) \leq 2$. Suppose finally that e is incident to a leaf of T. Then $\text{cutmim}_G(A_e, \overline{A_e}) = 1$. These observations imply that $\text{mimw}_G(T, \delta) \leq 2$. □

We can finally provide our two boundedness results. In both proofs, we make repeated implicit use of Ramsey's theorem: there exists a least positive integer $R(r, s)$ for which every graph with at least $R(r, s)$ vertices either contains an independent set of size r or a clique of size s. Observe that, for $r, s > 1$, $R(r, s) \geq s$.

Theorem 4.14. *Let $t \geq 4$ and let G be a $(3P_1, \overline{K_{3,t} + P_1})$-free graph. Then* $\text{mimw}(G) < 5R(3, t) + 8t + 46$ *and a branch decomposition (T, δ) of G with* $\text{mimw}_G(T, \delta) < 5R(3, t) + 8t + 46$ *can be constructed in $O(n^2)$ time.*

Proof. We assume that G contains two non-adjacent vertices v_a and v_b, or else G is a complete graph and the statement is trivially true. Let $S_z = \{v_a, v_b\}$. Since G is $3P_1$-free, all remaining vertices are adjacent to at least one of v_a and v_b and we partition them into three classes S_a, S_b and S_{ab} as follows: S_a is the set of vertices that are adjacent to v_a but not v_b, S_b is the set of vertices that are adjacent to v_b but not v_a and S_{ab} is the set of vertices that are adjacent to both v_a and v_b. Note that S_a is a clique, or else two non-adjacent vertices in S_a together with v_b would induce a copy of $3P_1$. Similarly, S_b is a clique.

We now proceed to the construction of a branch decomposition of G by distinguishing two cases. We say that G is *good* (w.r.t. $\{v_a, v_b\}$) if every vertex in S_a has at most two neighbours in S_b and every vertex in S_b has at most two neighbours in S_a. Otherwise, we say that G is *bad* (w.r.t. $\{v_a, v_b\}$).

Suppose first that G is good. Then, $G[S_a, S_b]$ has maximum degree at most 2 and, if $G[S_a, S_b]$ contains at least two vertices, Theorem 4.13 allows us to construct a branch decomposition (T_1, δ_1) of $G[S_a, S_b]$ with mim-width at most 2. Let u be a leaf of T_1 and let e be the edge of T_1 incident to u. We subdivide e by introducing a new vertex x and obtain a new tree T_1'. If however $G[S_a, S_b]$ contains exactly one vertex, let x be this vertex. We now let $\ell = |V(G) \setminus (S_a \cup S_b)|$ and consider an ℓ-caterpillar T_2 (notice that $\ell \geq 2$). We subdivide one of the edges of the backbone of T_2 by introducing a new vertex y and obtain a new tree T_2'. Let δ_2 be any bijection from $V(G) \setminus (S_a \cup S_b)$ to the set of leaves of T_2'. We finally add the edge xy in order to obtain a subcubic tree T, unless $G[S_a, S_b]$ is the null graph, in which case we let $T = T_2'$. Clearly, the set of leaves L of T is the disjoint union of the set of leaves of T_1 and the set of leaves of T_2. Considering the map $\delta\colon V(G) \to L$ which coincides with δ_1 when restricted to $S_a \cup S_b$ and with δ_2 when restricted to $V(G) \setminus (S_a \cup S_b)$, we obtain a branch decomposition (T, δ) of G. If G is bad, we simply let (T, δ) be any branch decomposition of G.

The branch decomposition (T, δ) of G defined above can be constructed in $O(n^2)$ time. Indeed, we first find two non-adjacent vertices v_a and v_b in $O(n^2)$ time and check whether $G[S_a, S_b]$ has maximum degree at most 2 in linear time. If so, G is good and we then construct (T, δ) in $O(n)$ time thanks to Theorem 4.13. Otherwise, G is bad, and we trivially construct (T, δ) in linear time.

Claim 4.15. *Let S_P and S_Q be subsets of vertices of G, not necessarily disjoint. If there exists a vertex that is complete to both S_P and S_Q, then* $\text{cutmim}_G(A_e \cap S_P, \overline{A_e} \cap S_Q) < R(3, t) + 6$, *for any $e \in E(T)$.*

Proof of Claim 4.15. Let $v \in V(G)$ be complete to S_P and S_Q. Suppose, to the contrary, that $\text{cutmim}_G(A_e \cap S_P, \overline{A_e} \cap S_Q) \geq R(3, t) + 6$ for some $e \in E(T)$ and let $\{p_1q_1, \ldots, p_{R(3,t)+6}q_{R(3,t)+6}\}$ be an induced matching witnessing this, where $\{p_1, \ldots, p_{R(3,t)+6}\} \subseteq A_e \cap S_P$ and $\{q_1, \ldots, q_{R(3,t)+6}\} \subseteq \overline{A_e} \cap S_Q$. Since G is $3P_1$-free, $\{q_1, \ldots, q_{R(3,t)}\}$ contains a clique of size at least t. Without loss of generality, $\{q_1, \ldots, q_t\}$ induces such a clique. Observe now that

$\{p_{R(3,t)+1}, \ldots, p_{R(3,t)+6}\}$ contains a clique of size 3, as $R(3,3) = 6$. Without loss of generality, $\{p_{R(3,t)+1}, p_{R(3,t)+2}, p_{R(3,t)+3}\}$ induces such a clique. But then we have that $G[p_{R(3,t)+1}, p_{R(3,t)+2}, p_{R(3,t)+3}, q_1, q_2, \ldots, q_t, v] \cong \overline{K_{3,t} + P_1}$, a contradiction. ⋄

Claim 4.16. *Suppose that G is bad. Then* $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_b) < 4t$ *and* $\text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_a) < 4t$*, for any $e \in E(T)$.*

Proof of Claim 4.16. By symmetry, it is enough to show the first statement. Since G is bad, $G[S_a, S_b]$ contains a vertex u of degree at least 3. Without loss of generality, $u \in S_a$. Suppose, to the contrary, that $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_b) \geq 4t$ for some $e \in E(T)$ and let $\{a_1b_1, \ldots, a_{4t}b_{4t}\}$ be an induced matching witnessing this, where $\{a_1, \ldots, a_{4t}\} \subseteq A_e \cap S_a$ and $\{b_1, \ldots, b_{4t}\} \subseteq \overline{A_e} \cap S_b$. Let $v_1, v_2, v_3 \in S_b$ be distinct neighbours of $u \in S_a$. Observe now that all except possibly $t-1$ vertices in $\{a_1, \ldots, a_{4t}\}$ are adjacent to at least one of v_1, v_2, v_3, or else there are t vertices in $\{a_1, \ldots, a_{4t}\}$, say without loss of generality $a_1, \ldots, a_t$, non-adjacent to any of v_1, v_2, v_3 and so, since S_a and S_b are cliques, $G[v_1, v_2, v_3, a_1, \ldots, a_t, u] \cong \overline{K_{3,t} + P_1}$, a contradiction. Hence, there is a vertex in $\{v_1, v_2, v_3\}$ with at least t neighbours in $\{a_1, \ldots, a_{4t}\}$, say without loss of generality v_1 is adjacent to $a_1, \ldots, a_t$, and so $G[b_{t+1}, b_{t+2}, b_{t+3}, a_1, \ldots, a_t, v_1] \cong \overline{K_{3,t} + P_1}$, a contradiction. ⋄

We can finally show that $\text{mimw}_G(T, \delta) < 5R(3,t) + 8t + 46$. Let $D = \{a, b, ab, z\}$. Since S_a, S_b, S_{ab}, S_z is a partition of $V(G)$, Observation 4.11 implies that

$$\text{mimw}_G(T, \delta) \leq \max_{e \in E(T)} \sum_{i,j \in D} \text{cutmim}_G(A_e \cap S_i, \overline{A_e} \cap S_j).$$

It is then enough to estimate the terms in the sum. Since S_a and S_b are cliques, $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_a) \leq 1$ and $\text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_b) \leq 1$. Moreover, since v_a is complete to S_a and S_{ab}, and v_b is complete to S_b and S_{ab}, Claim 4.15 implies that $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_{ab})$, $\text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_{ab})$, $\text{cutmim}_G(A_e \cap S_{ab}, \overline{A_e} \cap S_a)$, $\text{cutmim}_G(A_e \cap S_{ab}, \overline{A_e} \cap S_b)$, $\text{cutmim}_G(A_e \cap S_{ab}, \overline{A_e} \cap S_{ab}) < R(3,t) + 6$. Observe now that, for any $i \in D$, $\text{cutmim}_G(A_e \cap S_z, \overline{A_e} \cap S_i) \leq 2$, $\text{cutmim}_G(A_e \cap S_i, \overline{A_e} \cap S_z) \leq 2$.

It remains to bound $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_b)$ and $\text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_a)$. If G is bad then, by Claim 4.16, $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_b) < 4t$ and $\text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_a) < 4t$. If G is good, we proceed as follows. Suppose first that either $e = xy$ or $e \in E(T_2')$. Then all vertices of S_a and

S_b belong to the same partition class of $V(G)$ induced by e and so $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_b) = \text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_a) = 0$. Suppose finally that $e \in E(T_1')$. Then e induces a partition $(A_e', \overline{A_e'})$ of $S_a \cup S_b$ with respect to (T_1, δ_1), and $(A_e', \overline{A_e'})$ coincides with $(A_e, \overline{A_e})$ restricted to $S_a \cup S_b$. Consequently, $\text{cutmim}_G(A_e \cap S_a, \overline{A_e} \cap S_b) = \text{cutmim}_G(A_e' \cap S_a, \overline{A_e'} \cap S_b) \leq 2$ as $\text{cutmim}_G(T_1, \delta_1) \leq 2$. The same holds for $\text{cutmim}_G(A_e \cap S_b, \overline{A_e} \cap S_a)$.

By the previous paragraphs, $\text{mimw}_G(T, \delta) < 2 \cdot 1 + 5 \cdot (R(3,t) + 6) + 7 \cdot 2 + 2 \cdot 4t = 5R(3,t) + 8t + 46$. □

Theorem 4.17. *Let* $t \geq 4$ *and let* G *be a* $(4P_1, \overline{K_{2,t} + P_1})$*-free graph. Then* $\text{mimw}(G) < 43R(4,t) + 24t + 208$ *and a branch decomposition* (T, δ) *of* G *with* $\text{mimw}_G(T, \delta) < 43R(4,t) + 24t + 214$ *can be computed in* $O(n^3)$ *time.*

Proof. We assume that G contains three pairwise non-adjacent vertices v_a, v_b and v_c, or else G is $3P_1$-free and the statement follows from Theorem 4.14. Since G is $4P_1$-free, all remaining vertices are adjacent to at least one of v_a, v_b and v_c. For a subset $\alpha \subseteq \{a,b,c\}$, let $S_\alpha = \cap_{i \in \alpha} N(v_i) \setminus \cup_{j \in \{a,b,c\} \setminus \alpha} N(v_j)$. In words, S_α is the set of private neighbours of $\{v_i : i \in \alpha\}$ with respect to $\{v_a, v_b, v_c\}$. Note that S_a, S_b and S_c are cliques, or else, for distinct $i,j,k \in \{a,b,c\}$, two non-adjacent vertices in S_i together with v_j and v_k would induce a copy of $4P_1$. This fact will be repeatedly used in the claims below. For $\alpha, \beta \subseteq \{a,b,c\}$ and an integer $s \geq 1$, we say that the vertex set S_α is $3s$*-almost-complete* to the vertex set S_β if there are at most two vertices in S_α non-adjacent to at least $3s$ vertices in S_β.

Claim 4.18. *Let* $p, q \in \{a,b,c\}$ *with* $p \neq q$*. If a vertex in* S_p *is adjacent to at least two vertices in* S_q*, then* S_q *is* $3t$*-almost-complete to* S_p*.*

Proof of Claim 4.18. Note that v_p is complete to S_p but anticomplete to S_q and v_q is complete to S_q but anticomplete to S_p. Suppose that $x \in S_p$ is adjacent to two distinct vertices y_1 and y_2 of S_q. Then $\{y_1, y_2\} \cap \{v_q\} = \varnothing$.

We claim that there are at most $t-1$ vertices in S_p anticomplete to $\{y_1, y_2\}$. Indeed, if there are t vertices in S_p anticomplete to $\{y_1, y_2\}$, then these t vertices together with $\{x, y_1, y_2\}$ induce a copy of $\overline{K_{2,t} + P_1}$, as S_p and S_q are cliques, a contradiction.

Let now $y \in S_q$ be a vertex distinct from y_1 and y_2. We claim that y is anticomplete to at most $t-1$ vertices in $S_p \cap N(y_i)$, for each $i \in \{1,2\}$. Indeed, if there are t vertices in $S_p \cap N(y_i)$ anticomplete to y, then these t vertices together with $\{y_i, v_q, y\}$ induce a copy of $\overline{K_{2,t} + P_1}$, a contradiction.

Let $A_1 = S_p \cap N(y_1)$, $A_2 = S_p \cap N(y_2)$ and let $y \in S_q$ be a vertex distinct from y_1 and y_2. Clearly, $S_p = A_1 \cup A_2 \cup (S_p \setminus (A_1 \cup A_2))$. By the second paragraph, $|S_p \setminus (A_1 \cup A_2)| \leq t-1$ and so y is anticomplete to at most $t-1$ vertices in $S_p \setminus (A_1 \cup A_2)$. By the third paragraph, y is anticomplete to at most $t-1$ vertices in A_1 and at most $t-1$ vertices in A_2. Therefore, y is anticomplete to at most $3(t-1) < 3t$ vertices in S_p and so S_q is $3t$-almost-complete to S_p. $\diamond$

We now proceed to the construction of a branch decomposition of G. Consider first the graph G_1 with vertex set $V(G_1) = S_a \cup S_b \cup S_c$ and edge set $E(G_1) = \{uv : uv \in E(G), u \in S_\alpha, v \in S_\beta, \alpha, \beta \in \{a,b,c\}, \alpha \neq \beta, S_\alpha$ is not $3t$-almost-complete to S_β, S_β is not $3t$-almost complete to $S_\alpha\}$. We claim that each vertex v of G_1 has degree at most 2. By symmetry, suppose that $v \in S_a$. By definition of G_1, v has no neighbours in S_a. If S_b is $3t$-almost-complete to S_a, then v has no neighbours in S_b. Otherwise, S_b is not $3t$-almost-complete to S_a and, by Claim 4.18, v has at most one neighbour in S_b. Similarly, v has at most one neighbour in S_c. Therefore, G_1 has maximum degree at most 2 and so, by Theorem 4.13, if G_1 contains at least two vertices, then we can construct in $O(n)$ time a branch decomposition (T_1, δ_1) of G_1 with $\text{mimw}_{G_1}(T_1, \delta_1) \leq 2$.

For $x \in \{a,b,c\}$ and $Y = \{a,b,c\} \setminus \{x\}$, a vertex $v \in S_Y$ is S_x-*good* if it has at most one neighbour in S_x, and S_x-*bad* otherwise. Let $S_Y^\star$ be the set of vertices in S_Y that are S_x-bad. We now build a graph G_2 as follows. Start with $G_2 = G_1$. For each $x \in \{a,b,c\}$, let $Y = \{a,b,c\} \setminus \{x\}$. For each vertex $v \in S_Y$, if v is S_x-good, then add v to $V(G_2)$ and, if v has a neighbour u in S_x, add uv to $E(G_2)$. In other words, we grow G_1 by adding leaf vertices or isolated vertices.

Now, if G_2 is the null graph, let T_2' be the null tree, and if G_2 consists of one vertex, let T_2' be the tree with a single vertex r. Otherwise, G_2 contains at least two vertices and, given (T_1, δ_1), we can construct a branch decomposition (T_2, δ_2) of G_2 with $\text{mimw}_{G_2}(T_2, \delta_2) \leq 2$ in $O(n)$ time thanks to Theorem 4.12, unless G_1 contains at most one vertex, in which case G_2 has maximum degree at most 1 and we let (T_2, δ_2) be any branch decomposition of G_2. We then subdivide one of the edges of T_2 by introducing a new vertex r to obtain a new tree T_2'.

Clearly, $\text{mimw}_{G_2}(T_2', \delta_2) = \text{mimw}_{G_2}(T_2, \delta_2) \leq 2$. Let now $\ell = |V(G) \setminus V(G_2)|$ and consider an ℓ-caterpillar T_3 (notice that $\ell \geq 3$). Let δ_3 be any bijection from $V(G) \setminus V(G_2)$ to the set of leaves of T_3. We subdivide one of the edges of the backbone of T_3 by introducing a new vertex s and obtain a new tree T_3'. We finally add the edge rs in order to obtain a tree T. Observe that the set of leaves L of T is the disjoint union of the set of leaves L_2 of T_2' and the set of leaves L_3 of T_3'. Considering the map δ which coincides with δ_i when restricted to L_i (for $i = 2, 3$), we obtain a branch decomposition (T, δ) of G.

We now analyse the running time to construct (T, δ). Finding three pairwise non-adjacent vertices v_a, v_b and v_c and computing S_α for each $\alpha \subseteq \{a, b, c\}$ can be done in $O(n^3)$ time. Checking for $3t$-almost-completeness and constructing G_1 can be done in $O(n)$ time. Finding the S_x-good vertices and constructing G_2 can be done in $O(n)$ time. Therefore, constructing (T, δ) can be done in $O(n^3)$ time.

Claim 4.19. *Let $\alpha, \beta \subseteq \{a, b, c\}$. If S_α is $3t$-almost-complete to S_β, then* $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) < 3t + 1$ *and* $\text{cutmim}_G(A_e \cap S_\beta, \overline{A_e} \cap S_\alpha) < 3t + 1$*, for any $e \in E(T)$.*

Proof of Claim 4.19. Suppose that there exist $V_\alpha \subseteq A_e \cap S_\alpha$ and $V_\beta \subseteq \overline{A_e} \cap S_\beta$ such that $G[V_\alpha, V_\beta] \cong (3t + 1)P_2$. Then, each of the $3t + 1$ vertices in V_α is non-adjacent to at least $3t$ vertices in V_β, contradicting the fact that S_α is $3t$-almost-complete to S_β. The proof of the other inequality is similar. ◇

Claim 4.20. *Let $x \in \{a, b, c\}$ and $Y = \{a, b, c\} \setminus \{x\}$. Then* $\text{cutmim}_G(A_e \cap S_x, \overline{A_e} \cap S_Y^\star) < R(4, t) + t + 1$ *and* $\text{cutmim}_G(A_e \cap S_Y^\star, \overline{A_e} \cap S_x) < R(4, t) + t + 1$*, for any $e \in E(T)$.*

Proof of Claim 4.20. We show the first inequality, the proof of the other being similar. Suppose, to the contrary, that there exists $e \in E(T)$ such that $\text{cutmim}_G(A_e \cap S_x, \overline{A_e} \cap S_Y^\star) \geq R(4, t) + t + 1$. Let $\{p_1q_1, \ldots, p_{R(4,t)+t+1}q_{R(4,t)+t+1}\}$ be an induced matching witnessing this, where $P = \{p_1, \ldots, p_{R(4,t)+t+1}\} \subseteq S_x$ and $Q = \{q_1, \ldots, q_{R(4,t)+t+1}\} \subseteq S_Y^\star$. Since q_1 is S_x-bad, let $u_1 \in S_x$ be one of its neighbours distinct from p_1. Suppose that q_1 has at least $R(4, t)$ neighbours in Q. Then, at least t of these neighbours induce a clique. Without loss of generality, suppose that $\{q_2, \ldots, q_{t+1}\}$ are neighbours of q_1 inducing a clique. If $\{q_2, \ldots, q_{t+1}\}$ is anticomplete to u_1, then these t vertices together with $\{q_1, p_1, u_1\}$ induce a copy of $\overline{K_{2,t} + P_1}$, a contradiction.

Hence, u_1 has at least one neighbour in $\{q_2, \dots, q_{t+1}\}$, say without loss of generality q_2. But then, $G[q_1, q_2, p_3, \dots, p_{t+2}, u_1] \cong \overline{K_{2,t} + P_1}$, a contradiction.

Hence, q_1 has less than $R(4,t)$ neighbours in Q. Without loss of generality, suppose that $q_{R(4,t)+1}, \dots, q_{R(4,t)+t+1}$ are non-neighbours of q_1. Then, these $t+1$ vertices form a clique, or else two non-adjacent vertices v and v' among them would give $G[v, v', q_1, v_x] \cong 4P_1$, a contradiction. Next, since $q_{R(4,t)+1}$ is S_x-bad, it has another neighbour $u_2 \in S_x$ distinct from $p_{R(4,t)+1}$. Suppose that $\{q_{R(4,t)+2}, \dots, q_{R(4,t)+t+1}\}$ is anticomplete to u_2. Then, we have that $G[p_{R(4,t)+1}, u_2, q_{R(4,t)+2}, \dots, q_{R(4,t)+t+1}, q_{R(4,t)+1}] \cong \overline{K_{2,t} + P_1}$, a contradiction. Therefore, u_2 has at least one neighbour in $\{q_{R(4,t)+2}, \dots, q_{R(4,t)+t+1}\}$, say without loss of generality $q_{R(4,t)+2}$. Then, $G[q_{R(4,t)+1}, q_{R(4,t)+2}, p_1, \dots, p_t, u_2] \cong \overline{K_{2,t} + P_1}$, a contradiction. $\diamond$

Claim 4.21. *Let $\alpha, \beta \subseteq \{a, b, c\}$ with $\alpha \cap \beta \neq \varnothing$. Then* $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) < R(4,t) + 4$, *for any $e \in E(T)$.*

Proof of Claim 4.21. Let $i \in \alpha \cap \beta$. Then v_i is complete to S_α and S_β. Suppose, to the contrary, that there exists $e \in E(T)$ such that $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) \geq R(4,t) + 4$. Let $\{p_1q_1, \dots, p_{R(4,t)+4}q_{R(4,t)+4}\}$ be an induced matching witnessing this, where $P = \{p_1, \dots, p_{R(4,t)+4}\} \subseteq S_\alpha$ and $Q = \{q_1, \dots, q_{R(4,t)+4}\} \subseteq S_\beta$. Since G is $4P_1$-free, Q contains a clique of size at least t. Without loss of generality, suppose that $\{q_1, \dots, q_t\}$ induces a clique. Observe now that $\{p_{R(4,t)+1}, \dots, p_{R(4,t)+4}\}$ contains a pair of adjacent vertices, as G is $4P_1$-free. Without loss of generality, suppose that $p_{R(4,t)+1}$ is adjacent to $p_{R(4,t)+2}$. But then, $G[p_{R(4,t)+1}, p_{R(4,t)+2}, q_1, q_2, \dots, q_t, v_i] \cong \overline{K_{2,t} + P_1}$, a contradiction. $\diamond$

We can finally show that $\text{mimw}_G(T, \delta) < 43R(4,t) + 24t + 214$. Let $S_z = \{v_a, v_b, v_c\}$ and let $D = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}, \{z\}\}$. Since $\{S_\alpha : \alpha \in D\}$ is a partition of $V(G)$, Observation 4.11 implies that

$$\text{mimw}_G(T, \delta) \leq \max_{e \in E(T)} \sum_{\alpha, \beta \in D} \text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta). \tag{4.2}$$

For any $\alpha \in D$, we have that $\text{cutmim}_G(A_e \cap S_z, \overline{A_e} \cap S_\alpha) \leq 3$ and $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_z) \leq 3$, for any $e \in E(T)$. For $\alpha, \beta \neq \{z\}$, there are 49 distinct pairs (α, β). 12 of such pairs are such that $\alpha \cap \beta = \varnothing$: $(\{a\}.\{b\}), (\{b\}, \{c\}), (\{a\}, \{c\}), (\{a\}, \{b,c\}), (\{b\}, \{a,c\}), (\{c\}, \{a,b\})$ and those obtained by swapping α and β. The remaining 37 pairs are such that $\alpha \cap \beta \neq \varnothing$. In this case, by Claim 4.21, $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) \leq R(4,t) + 4$, for any $e \in E(T)$.

We now estimate the terms in the sum above corresponding to pairs (α, β) such that $\alpha \cap \beta = \varnothing$. Suppose first that (α, β) is one of $(\{a\}, \{b\}), (\{b\}, \{c\}), (\{a\}, \{c\}), (\{b\}, \{a\}), (\{c\}, \{b\}), (\{c\}, \{a\})$. If S_α is $3t$-almost-complete to S_β or S_β is $3t$-almost-complete to S_α then, by Claim 4.19, $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) < 3t + 1$ and $\text{cutmim}_G(A_e \cap S_\beta, \overline{A_e} \cap S_\alpha) < 3t + 1$. Otherwise, S_α is not $3t$-almost-complete to S_β and S_β is not $3t$-almost-complete to S_α. By definition of G_1 and G_2, this implies that $G[S_\alpha, S_\beta] = G_1[S_\alpha, S_\beta] = G_2[S_\alpha, S_\beta]$. If either $e = rs$ or e belongs to T_3', then all vertices of S_α and S_β belong to the same partition class of $V(G)$ induced by e and so $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) = \text{cutmim}_G(A_e \cap S_\beta, \overline{A_e} \cap S_\alpha) = 0$. Otherwise, e must belong to T_2'. The edge e then induces a partition $(A_e', \overline{A_e'})$ of the vertices of G_2 with respect to (T_2', δ_2), and $(A_e', \overline{A_e'})$ coincides with $(A_e, \overline{A_e})$ restricted to $S_\alpha \cup S_\beta$. Hence, $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) = \text{cutmim}_{G_2}(A_e' \cap S_\alpha, \overline{A_e'} \cap S_\beta) \leq 2$.

Suppose finally that (α, β) is one of $(\{a\}, \{b, c\})$, $(\{b\}, \{a, c\})$, $(\{c\}, \{a, b\})$, $(\{b, c\}, \{a\})$, $(\{a, c\}, \{b\})$, $(\{a, b\}, \{c\})$. Clearly, $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) \leq \text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta^\star) + \text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap (S_\beta \setminus S_\beta^\star))$. Note that $G[S_\alpha, S_\beta \setminus S_\beta^\star] = G_2[S_\alpha, S_\beta \setminus S_\beta^\star]$. Thus, by the same reasoning as in the previous paragraph, $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap (S_\beta \setminus S_\beta^\star)) \leq 2$. On the other hand, by Claim 4.20, $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta^\star) \leq R(4,t) + t + 1$. Therefore, $\text{cutmim}_G(A_e \cap S_\alpha, \overline{A_e} \cap S_\beta) \leq R(4,t) + t + 3$.

Combining these bounds with (4.2), we obtain $\text{mimw}_G(T, \delta) < 14 \cdot 3 + 37 \cdot (R(4,t) + 4) + 6 \cdot (3t + 1) + 6 \cdot (R(4,t) + t + 3) = 43R(4,t) + 24t + 214$. □

4.2.2 Unboundedness results

All the unboundedness results in this section are obtained by applying the same strategy. The class of walls plays a crucial role. A *wall of height h and width r* (an *$h \times r$-wall* for short) is the graph that can be obtained from the grid of height h and width $2r$ as follows. Let $C_1, \ldots, C_{2r}$ be the set of vertices in each of the $2r$ columns of the grid, ordered from left to right. For each column C_j, let $e_1^j, e_2^j, \ldots, e_{h-1}^j$ be the edges between two vertices of C_j, in top-to-bottom order. If j is odd, delete all edges e_i^j with i even. If j is even, delete all edges e_i^j with i odd.

Figure 4.3: An elementary wall of height 2 and a net-wall.

By removing all degree-1 vertices of the resulting graph, we obtain an *elementary* $h \times r$*-wall*. Finally, an $h \times r$-wall is any subdivision of the elementary $h \times r$-wall. See Figure 4.3 for a small example. A *net-wall* is a graph obtained from a wall G by replacing every vertex u of G that has three distinct neighbours v, w, x by three new vertices u_v, u_w, u_x and edges $u_v v$, $u_w w$, $u_x x$, $u_v u_w$, $u_v u_x$, $u_w u_x$. See Figure 4.3.

Theorem 4.22 (Brettell et al. [26])**.** *Let W be an elementary $n \times n$ wall with $n \geq 7$. Then* $\text{mimw}(W) \geq \frac{\sqrt{n}}{50}$*. Hence, $\mathcal{W}$ has unbounded mim-width.*

The idea is to start from an elementary wall, find an appropriate vertex colouring, and repeatedly apply the following result (the case $k = 2$ was first proved in [129]).

Lemma 4.23 (Brettell et al. [26])**.** *Let G be a k-partite graph with partition classes $V_1, \ldots, V_k$ and let G' be a graph obtained from G by adding edges where, for each added edge, there exists some i such that both endpoints are in V_i. Then* $\text{mimw}(G') \geq \frac{1}{k} \cdot \text{mimw}(G)$.

Theorem 4.24. *The class of* $(3P_1, \overline{K_{4,4} + P_1})$*-free graphs has unbounded mim-width.*

Proof. Let W be an elementary $2n \times 2n$ wall and consider its proper 2-colouring with red and blue as depicted in Figure 4.4. We add edges within each colour class to make them cliques. Let $f(W)$ be the graph obtained and let $\mathcal{W}_1 = \{f(W) : W \in \mathcal{W}\}$. By Theorem 4.22 and Lemma 4.23, $\mathcal{W}_1$ has unbounded mim-width.

Note that, for the graph $f(W)$, every two vertices of the same colour are adjacent, and every two vertices of different colours are adjacent if and only if they are adjacent in W. Clearly, $f(W)$ is $3P_1$-free. It remains to show that $f(W)$ is $\overline{K_{4,4} + P_1}$-free.

Claim 4.25. *Any copy of K_5 in $f(W)$ is monochromatic.*

Proof of Claim 4.25. Let $u_1, \ldots, u_5$ be the vertices of a copy of K_5. Since $f(W)$ is obtained from W by adding edges within each colour class, if an edge $uv \in E(f(W))$ is not monochromatic,

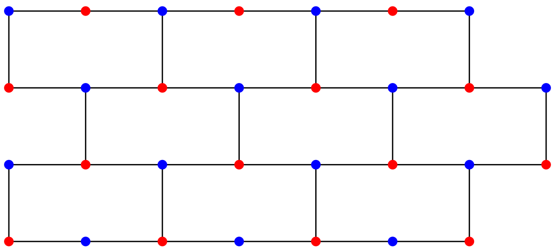

Figure 4.4: A 2-colouring of the elementary 4×4 wall.

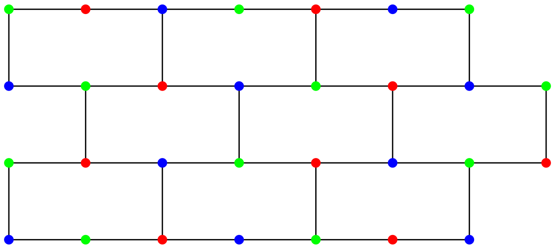

Figure 4.5: A 3-colouring of the elementary 4×4 wall.

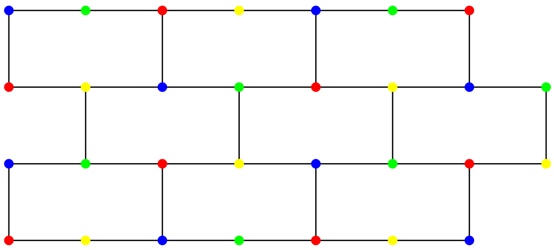

Figure 4.6: A 4-colouring of the elementary 4×4 wall.

then uv belongs to $E(W)$ as well. Hence, there cannot be one blue vertex and four red vertices in $\{u_1, \ldots, u_5\}$, since this would imply that in W there is a vertex with four neighbours. Also, there cannot be exactly two blue vertices in $\{u_1, \ldots, u_5\}$, for otherwise these two blue vertices share three common red neighbours in W, contradicting the fact that in W any two vertices have at most one common neighbour. By symmetry, there cannot be exactly one or two red vertices, and so $\{u_1, \ldots, u_5\}$ is monochromatic. ◇

Suppose, to the contrary, that $f(W)$ contains an induced copy of $\overline{K_{4,4} + P_1}$ with vertex set $\{v_0, \ldots, v_8\}$ as depicted in Figure 4.1. By Claim 4.25, the two copies of K_5 induced by $\{v_0, v_1, v_2, v_3, v_4\}$ and $\{v_0, v_5, v_6, v_7, v_8\}$ must both be monochromatic. Hence, $v_1, \ldots, v_8$ must be of the same colour. This implies that $v_1, \ldots, v_8$ must form a clique in $f(W)$, a contradiction. □

Theorem 4.26. *The class of* $(4P_1, \overline{K_{3,3} + P_1})$*-free graphs has unbounded mim-width.*

Proof. Let W be an elementary $2n \times 2n$ wall and consider its proper 3-colouring depicted in Figure 4.5. We add edges within each colour class to make them cliques. Let $g(W)$ be the graph obtained and let $\mathcal{W}_2 = \{g(W) : W \in \mathcal{W}\}$. By Theorem 4.22 and Lemma 4.23, $\mathcal{W}_2$ has unbounded mim-width. Clearly, $g(W)$ is $4P_1$-free. It remains to show that $g(W)$ is $\overline{K_{3,3} + P_1}$-free.

Claim 4.27. *Any copy of K_4 in $g(W)$ is monochromatic.*

Proof of Claim 4.27. Let $u_1, \ldots, u_4$ be the vertices of a copy of K_4. At least two such vertices have the same colour, say colour c. Since $g(W)$ is obtained from W by adding edges within each colour class, if an edge $uv \in E(g(W))$ is not monochromatic, then uv belongs to $E(W)$ as well. Observe first that there cannot be exactly two vertices with colour c in $\{u_1, \ldots, u_4\}$, for otherwise these two vertices coloured c have two common neighbours coloured different from c, contradicting the fact that in W any two vertices have at most one common neighbour. Moreover, there cannot be exactly three vertices coloured c in $\{u_1, \ldots, u_4\}$, since this would imply that in W there is a vertex not coloured c adjacent to three vertices coloured c. However, in the 3-colouring of W depicted in Figure 4.5, no vertex has three monochromatic neighbours. ◇

Suppose, to the contrary, that $g(W)$ contains an induced copy of $\overline{K_{3,3} + P_1}$ with vertex set $\{v_0, \ldots, v_6\}$, where v_0 is the universal vertex and $\{v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6\}$ induce disjoint cliques. By Claim 4.27, the two copies of K_4 induced by $\{v_0, v_1, v_2, v_3\}$ and $\{v_0, v_4, v_5, v_6\}$ must both be monochromatic. Hence, $v_1, \ldots, v_6$ must be of the same colour. This implies that $v_1, \ldots, v_6$ must form a clique in $g(W)$, a contradiction. □

Theorem 4.28. *The class of $(5P_1, \overline{K_{2,2} + P_1})$-free graphs has unbounded mim-width.*

Proof. Let W be an elementary $2n \times 2n$ wall and consider its proper 4-colouring depicted in Figure 4.6. We add edges within each colour class to make them cliques. Let $h(W)$ be the graph obtained and let $\mathcal{W}_3 = \{h(W) : W \in \mathcal{W}\}$. By Theorem 4.22 and Lemma 4.23, $\mathcal{W}_3$ has unbounded mim-width. Clearly, $h(W)$ is $5P_1$-free. It remains to show that $h(W)$ is $\overline{K_{2,2} + P_1}$-free.

Claim 4.29. *Any copy of K_3 in $h(W)$ is monochromatic.*

Proof of Claim 4.29. Let u_1, u_2, u_3 be the vertices of a copy of K_3. Firstly, there cannot be exactly two vertices in $\{u_1, u_2, u_3\}$ of the same colour, say colour c, since this would imply that in W there is a vertex coloured different from c which is adjacent to two vertices coloured c, contradicting the 4-colouring of W depicted in Figure 4.6. Moreover, the vertices in $\{u_1, u_2, u_3\}$ cannot be coloured with distinct colours, for otherwise these three vertices would induce a K_3 in W. ◇

Similarly to Theorems 4.24 and 4.26, it is now easy to see that $h(W)$ is $\overline{K_{2,2} + P_1}$-free. □

4.3 Mim-width of $(K_r, sP_1 + tP_2 + uP_3)$-free graphs

In this section we address Question 2 and prove Theorem 4.9 and Theorem 4.10. Both results are obtained by identifying new $(K_r, sP_1 + tP_2 + uP_3)$-free classes of unbounded mim-width.

Question 2 was formulated in [27] starting from [27, Theorem 35]. We remark that there is a typo in the formulation of this statement. For completeness we provide the correct formulation, whose proof is essentially identical to that of [27, Theorem 35].

Theorem 4.30 (Brettell et al. [26])**.** *Let H be a graph and let $r \geq 4$ be an integer. Let $\mathcal{S}$ be the class of graphs every component of which is either a subdivided claw or a path. Then exactly one of the following holds:*

- *$H \subseteq_i sP_1 + P_5$ or tP_2, and the mim-width of the class of (K_r, H)-free graphs is bounded and quickly computable;*
- *$H \notin \mathcal{S}$, or $H \supseteq_i K_{1,3}$, $P_2 + P_4$, or P_6, and the mim-width of the class of (K_r, H)-free graphs is unbounded; or*
- *$H = sP_1 + tP_2 + uP_3$, where $u \geq 1$ and $t + u \geq 2$.*

Proof. By [26, Theorem 31-(i)], if $H \notin \mathcal{S}$, then the mim-width of the class of (K_r, H)-free graphs is unbounded. So we may assume that H is a forest of paths and subdivided claws. By [26, Theorem 31-(iii)], if H contains a $K_{1,3}$, then the mim-width is again unbounded. So we may assume that H is a linear forest. If $H \subseteq_i sP_1 + P_5$ or $H \subseteq_i tP_2$, then mim-width is bounded and quickly computable by parts (xii) and (xiv) of [26, Theorem 30]. So we may assume that H is a linear forest containing $P_2 + P_3$. By [26, Theorem 31-(viii)], we may also assume H contains neither $P_2 + P_4$ nor P_6, otherwise the mim-width is again unbounded. It now follows that $H \subseteq_i tP_2 + uP_3$ for some u, t such that $u \geq 1$ and $t + u \geq 2$. Therefore, $H = sP_1 + tP_2 + uP_3$, for $u \geq 1$ and $t + u \geq 2$. □

4.3.1 Unboundedness results

Similarly to Section 4.2.2, the unboundedness results for $(K_r, sP_1 + tP_2 + uP_3)$-free graphs in this section (Theorem 4.33 for $r = 5$ and Theorem 4.36 for $r = 4$) are obtained by applying Lemma 4.23. However, in the case of Theorem 4.36, only certain types of edges are added inside each colour class; this is to avoid creating copies of K_4. We will also make use of the following two results.

Lemma 4.31 (Vatshelle [150]). *Let G be a graph and $v \in V(G)$. Then* $\text{mimw}(G) \geq \text{mimw}(G - v)$.

Lemma 4.32 (Brettell et al. [26]). *Let G be a graph and let G' be the graph obtained by 1-subdividing an edge of G. Then* $\text{mimw}(G') \geq \text{mimw}(G)$.

Theorem 4.33. *The class of $(K_5, P_3 + P_2 + P_1)$-free graphs has unbounded mim-width.*

Proof. Consider first a $2n \times 2n$-grid G_{2n} with vertex set $\{(i, j) : 1 \leq i, j \leq 2n\}$. Consider the set of vertices $S = \{(i, j) : i + j \equiv 1 \pmod 2\}$ and the set of edges $T = \{(i, j)(i, j - 1) : (i, j) \in S\}$. We define the graph W_n as $W_n = (V(G_n), E(G_n) \setminus T)$. Since W_n contains the elementary $n \times n$ wall as an induced subgraph, Theorem 4.22 and Lemma 4.31 imply that the class of graphs $\{W_n : n \geq 1\}$ has unbounded mim-width. Given W_n, we now consider the following partition of its vertices:

$$A = \{(i,j) : i+j \equiv 0 \pmod 2,\ i \equiv 1 \pmod 3\}$$
$$B = \{(i,j) : i+j \equiv 0 \pmod 2,\ i \equiv 2 \pmod 3\}$$
$$C = \{(i,j) : i+j \equiv 0 \pmod 2,\ i \equiv 0 \pmod 3\}$$
$$D = \{(i,j) : i+j \equiv 1 \pmod 2,\ i \equiv 1 \pmod 3\}$$
$$E = \{(i,j) : i+j \equiv 1 \pmod 2,\ i \equiv 2 \pmod 3\}$$
$$F = \{(i,j) : i+j \equiv 1 \pmod 2,\ i \equiv 0 \pmod 3\}.$$

We then colour in red the vertices in $A \cup B \cup C$, and in blue the vertices in $D \cup E \cup F$ (see Figure 4.7). This gives a proper 2-colouring of W_n and, in particular, each partition class defined above forms an independent set in W_n. Observe that each vertex is adjacent to at most one vertex from each partition class of the opposite colour. That is, each vertex in $A \cup B \cup C$ is adjacent to at most one vertex from each of D, E and F, and each vertex in $D \cup E \cup F$ is adjacent to at most one vertex from each of A, B and C.

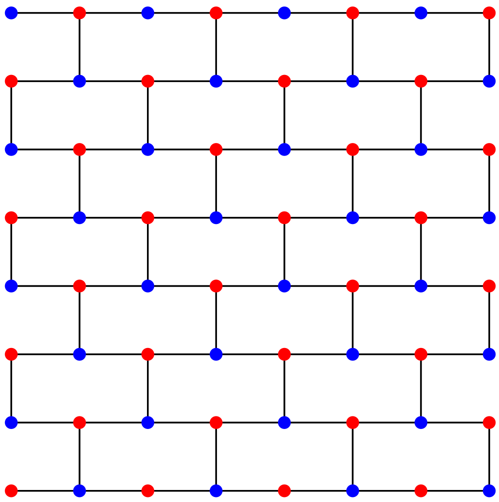

Figure 4.7: The graph W_4 with the red-blue colouring as in the proof of Theorem 4.33.

We now build the graph W'_n by adding all edges between different partition classes of the same colour. That is, we make A, B, C pairwise complete and D, E, F pairwise complete. No other edges are added. In particular, $W'_n[A \cup B \cup C]$ and $W'_n[D \cup E \cup F]$ are complete tripartite graphs.

Applying Lemma 4.23 to the the bipartition $(A \cup B \cup C, D \cup E \cup F)$ of $V(W_n)$, we obtain that $\text{mimw}(W'_n) \geq \text{mimw}(W_n)/2$. Hence, the class of graphs $\{W'_n : n \geq 1\}$ has unbounded mim-width. It is then enough to show that W'_n is K_5-free and $(P_3 + P_2 + P_1)$-free.

Claim 4.34. *W'_n is K_5-free.*

Proof of Claim 4.34. Suppose, to the contrary, that $\{v_1, v_2, v_3, v_4, v_5\}$ induces a copy of K_5 in W'_n. Since each of A, B, C, D, E, F is an independent set, the v_i's belong to different partition classes. In particular, without loss of generality, v_1, v_2, v_3 are red and v_4, v_5 blue, or vice versa. Since no edges between red and blue vertices are added when constructing W'_n, we have that $\{v_1, v_2, v_3\}$ is complete to $\{v_4, v_5\}$ in W_n. But this contradicts the fact that in W_n no two vertices have two common neighbours. ◇

Claim 4.35. *W'_n is $(P_3 + P_2 + P_1)$-free.*

Proof of Claim 4.35. Suppose, to the contrary, that $\{v_1, \ldots, v_6\}$ induces a copy of $P_3 + P_2 + P_1$ in W'_n, where $W'_n[v_1, v_2, v_3] \cong P_3$ with v_2 adjacent to both v_1 and v_3, $\{v_4, v_5\}$ is anticomplete to $\{v_1, v_2, v_3\}$ and induces a copy of P_2, and $\{v_6\}$ is anticomplete to $\{v_1, \ldots, v_5\}$. Suppose, without loss of generality, that v_2 is red.

Case 1: Both v_1 and v_3 are blue. Since v_4 is adjacent to at most one vertex from each blue partition class, we have that v_1 and v_3 belong to different blue partition classes. By construction, these partition classes are complete, contradicting the fact that v_1 is non-adjacent to v_3 in W'_n.

Case 2: At least one of v_1 and v_3 is red. Without loss of generality, v_1 is red. Since each partition class forms an independent set in W'_n, we have that v_1 does not belong to the class of v_2. But then $\{v_1, v_2\}$ dominates the red vertices and so v_4, v_5, v_6 are all blue. By a similar reasoning, v_4, v_5, v_6 all belong to the same blue partition class, or else there exists a vertex in $\{v_4, v_5, v_6\}$ dominating the remaining two. But each partition class is an independent set, contradicting the fact that v_4 is adjacent to v_5. ◇

This concludes the proof of Theorem 4.33. □

Theorem 4.36. *The class of $(K_4, P_3{+}2P_2{+}P_1, 2P_3{+}P_2)$-free graphs has unbounded mim-width.*

Proof. Let W_n be the graph defined in the proof of Theorem 4.33. Given W_n, we subdivide every edge $(i_1, j_1)(i_2, j_2)$ by adding a new vertex $(\frac{i_1+i_2}{2}, \frac{j_1+j_2}{2})$. We then multiply the coordinates of all vertices by 2 (so, e.g., $(4, 5.5)$ becomes $(8, 11)$) and preserve the adjacencies between vertices in order to obtain a new graph W'_n. By Lemma 4.32, $\text{mimw}(W'_n) \geq \text{mimw}(W_n)$. We now define a partition of the vertices of W'_n as follows (see Figure 4.8):

$$X = \{(i, j) : i + j \equiv 2 \pmod 4\}$$
$$Y = \{(i, j) : i + j \equiv 0 \pmod 4\}$$
$$A = \{(i, j) : i + j \text{ is odd}, \; i \equiv 1 \pmod 3\}$$
$$B = \{(i, j) : i + j \text{ is odd}, \; i \equiv 2 \pmod 3\}$$
$$C = \{(i, j) : i + j \text{ is odd}, \; i \equiv 0 \pmod 3\}$$

Note that X and Y consist of the vertices of W_n, and A, B and C consist of the new vertices introduced after edge subdivisions. In particular, each partition class is an independent set. Moreover, X is anticomplete to Y, and A, B, C are pairwise anticomplete. Since W'_n has no cycle of length 4, each $x \in X$ and $y \in Y$ have at most one common neighbour in $A \cup B \cup C$.

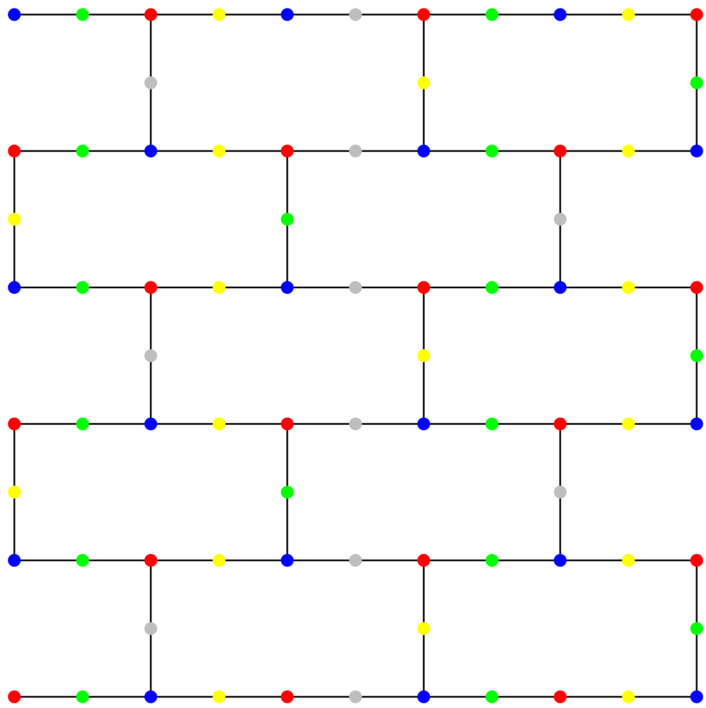

Figure 4.8: The graph W'_3 in the proof of Theorem 4.36, together with a proper 5-colouring: blue vertices correspond to X, red vertices to Y, grey vertices to A, yellow vertices to B and green vertices to C.

Observation 4.37. *Let $u_1 = (i_1, j_1)$ and $u_2 = (i_2, j_2)$ be two vertices belonging to the same partition class in $\{A, B, C\}$. The following hold:*

- 3 *divides* $|i_1 - i_2|$;
- *If* $i_1 = i_2$, *then* 2 *divides* $|j_1 - j_2|$.

We now proceed to the construction of the graph W''_n, obtained as follows. Start from W'_n and

- Add all edges between X and Y;
- For each pair of distinct sets R and S in $\{A, B, C\}$ and $r = (i_r, j_r) \in R$ and $s = (i_s, j_s) \in S$, add the edge rs, unless $j_r = j_s$ and $|i_r - i_s| = 2$, that is, unless r and s are the "right neighbour" and the "left neighbour" of a vertex in $X \cup Y$.

The edges left out in the second step above avoid the creation of copies of K_4, as will be shown shortly.

Since $(X \cup Y, A \cup B \cup C)$ is a bipartition of $V(W''_n)$, Lemma 4.23 implies that $\text{mimw}(W''_n) \geq \text{mimw}(W'_n)/2$. Hence the class of graphs $\{W''_n : n \geq 1\}$ has unbounded mim-width. It is then enough to show that W''_n does not contain any graph in $\{K_4, P_3 + 2P_2 + P_1, 2P_3 + P_2\}$ as an induced subgraph. This will be done in a series of claims.

Claim 4.38. *W''_n is K_4-free.*

Proof of Claim 4.38. Suppose, to the contrary, that $\{v_1, v_2, v_3, v_4\}$ induces a copy of K_4 in W''_n. Since each of X, Y, A, B and C is an independent set, the four vertices belong to four different partition classes.

Suppose first that exactly one of $\{v_1, v_2, v_3, v_4\}$ belongs to $X \cup Y$. Without loss of generality, $v_1 \in X$ and $v_2, v_3, v_4 \in A \cup B \cup C$. Since no edges between X and $A \cup B \cup C$ are added to $E(W'_n)$ in order to build W''_n, the vertices v_2, v_3, v_4 are adjacent to v_1 in W'_n. Suppose that $v_1 = (i, j)$. Then, up to relabelling, we must have that $v_2 = (i - 1, j), v_3 = (i + 1, j)$ and $v_4 = (i, j \pm 1)$. In other words, v_2 and v_3 are the left neighbour and right neighbour of v_1, respectively. But then, by construction, $v_2v_3 \notin E(W''_n)$, a contradiction.

Suppose finally that exactly two vertices of $\{v_1, v_2, v_3, v_4\}$ belong to $X \cup Y$. Without loss of generality, $v_1, v_2 \in X \cup Y$, and $v_3, v_4 \in A \cup B \cup C$. Since no edges between $X \cup Y$ and $A \cup B \cup C$ are added to $E(W'_n)$ in order to build W''_n, both v_1 and v_2 are adjacent to v_3 and v_4 in W'_n, contradicting the fact that W'_n does not contain any cycle of length 4. ⋄

Claim 4.39. *Let u_1, u_2 be two distinct vertices from the same partition class in $\{A, B, C\}$. Let u_3 be a vertex from a partition class in $\{A, B, C\}$ different from that of u_1 and u_2. Then u_3 is adjacent to at least one of u_1 and u_2.*

Proof of Claim 4.39. Let $u_1 = (i_1, j_1)$, $u_2 = (i_2, j_2)$ and $u_3 = (i_3, j_3)$. Suppose, to the contrary, that u_3 is non-adjacent to both u_1 and u_2. By construction of W''_n, this implies that $j_1 = j_3 = j_2$ and $|i_1 - i_3| = 2 = |i_2 - i_3|$. Since u_1 and u_2 are distinct, $i_1 \neq i_2$, which implies that $|i_1 - i_2| = 4$, contradicting the first part of Observation 4.37. ⋄

We now prove that W''_n is $(P_3 + 2P_2 + P_1)$-free and $(2P_3 + 2P_2)$-free. The following result will be used as the backbone of both proofs.

Claim 4.40. *Suppose that $\{v_1, \ldots, v_7\}$ induces a copy of $P_3 + 2P_2$, where v_2 is adjacent to v_1 and v_3, v_4 is adjacent to v_5, v_6 is adjacent to v_7 and no other edges are present in $W''_n[\{v_1, \ldots, v_7\}]$. Then the following hold:*

1. *At least one of v_1 and v_3 belongs to $X \cup Y$;*
2. *$v_2 \in X \cup Y$.*

Proof of Claim 4.40. We first show that at least one of v_1 and v_3 belongs to $X \cup Y$. Suppose, to the contrary, that both v_1 and v_3 belong to $A \cup B \cup C$. Since A, B and C are pairwise disjoint, $v_1, v_3 \in S \cup T$ for some distinct $S, T \in \{A, B, C\}$.

Observe that at least two of v_4, v_5, v_6, v_7, say v_i and v_j, belong to $A \cup B \cup C$, or else at least three vertices among v_4, v_5, v_6, v_7 belong to $X \cup Y$ and so $W''_n[X \cup Y]$ contains a copy of $P_2 + P_1$, contradicting the fact that $W''_n[X \cup Y]$ is a complete bipartite graph.

Observe now that, by Claim 4.39 and the previous paragraph, v_1 and v_3 belong the same partition class. Without loss of generality, $v_1, v_3 \in S$. Since S is an independent set, $v_2 \notin S$, and since each vertex in $X \cup Y$ has at most one neighbour in each of A, B and C, we have that $v_2 \notin X \cup Y$. Moreover, by Claim 4.39, v_i and v_j both belong to S. But this contradicts Claim 4.39, as $v_2 \in (A \cup B \cup C) \setminus S$.

We finally show that $v_2 \in X \cup Y$. Suppose, to the contrary, that $v_2 \in R$, for some $R \in \{A, B, C\}$. Since R is an independent set, $v_1, v_3 \notin R$. In view of part 1, we distinguish two cases, according to which partition classes v_1 and v_3 belong. Let S and T be the two distinct partition classes in $\{A, B, C\} \setminus R$.

Case 1: v_1 and v_3 both belong to $X \cup Y$.

Since each vertex in R is adjacent to at most one vertex in X and at most one vertex in Y, one of v_1 and v_3 belongs to X and the other to Y, contradicting the fact that X is complete to Y.

Case 2: One of v_1 and v_3 belongs to $X \cup Y$ and the other to $S \cup T$.

Without loss of generality, $v_1 \in S$ and $v_3 \in X$. Since X is complete to Y, $v_4, v_5, v_6, v_7 \notin Y$. Since X is an independent set, at most one of v_4 and v_5 belongs to X and at most one of v_6 and v_7 belongs to X. Without loss of generality, $v_4, v_6 \in A \cup B \cup C$. If both v_4 and v_6 belong to R, then $v_1 \in S$ is non-adjacent to both $v_4, v_6 \in R$, contradicting Claim 4.39. If exactly one of v_4 and v_6 belongs to R, say without loss of generality $v_4 \in R$ and $v_6 \notin R$, then $v_6 \in (A \cup B \cup C) \setminus R$ is non-adjacent to $v_2 \in R$ and $v_4 \in R$, contradicting Claim 4.39. Therefore, none of v_4 and v_6 belongs to R. If v_4 and v_6 belong to the same partition class in $(A \cup B \cup C) \setminus R$, then $v_2 \in R$ being non-adjacent to both of them contradicts Claim 4.39. Finally, if v_4 and v_6 belong to different partition classes in $(A \cup B \cup C) \setminus R$, then one of them belongs to the partition class S of v_1, say without loss of generality $v_4 \in S$. But then v_6 being non-adjacent to both v_1 and v_4 contradicts Claim 4.39. $\diamond$

Claim 4.41. *W''_n is $(P_3 + 2P_2 + P_1)$-free.*

Proof of Claim 4.41. Suppose, to the contrary, that $\{v_1, \ldots, v_8\}$ induces a copy of $P_3 + 2P_2 + P_1$, where v_2 is adjacent to v_1 and v_3, v_4 is adjacent to v_5, v_6 is adjacent to v_7 and no other edges are present in $W''_n[\{v_1, \ldots, v_8\}]$ (hence v_8 is the isolated vertex). By Claim 4.40, $v_2 \in X \cup Y$ and at least one of v_1 and v_3 belongs to $X \cup Y$. Without loss of generality, $v_2 \in X$ and $v_1 \in X \cup Y$. Since X is an independent set, $v_1 \in Y$. Since X is complete to Y, we have that $\{v_1, v_2\}$ dominates

$X \cup Y$ and so $\{v_4, \ldots, v_8\} \subseteq A \cup B \cup C$. By the pigeonhole principle, there exists two vertices among v_4, v_5, v_6, v_7 that belong to the same partition class in $\{A, B, C\}$. Since these classes form independent sets, the two vertices are non-adjacent. Without loss of generality, $v_4, v_6 \in R$ for some $R \in \{A, B, C\}$. If $v_8 \in (A \cup B \cup C) \setminus R$, then v_8 is non-adjacent to both $v_4, v_6 \in R$, contradicting Claim 4.39. Therefore, $v_8 \in R$. Since R is an independent set, $v_4 \in R$ implies that $v_5 \in (A \cup B \cup C) \setminus R$ and v_5 is non-adjacent to both $v_6, v_8 \in R$, contradicting Claim 4.39. ◇

Claim 4.42. *W''_n is $(2P_3 + P_2)$-free.*

Proof of Claim 4.42. Suppose, to the contrary, that $\{v_1, \ldots, v_8\}$ induces a copy of $2P_3 + P_2$, where v_2 is adjacent to v_1 and v_3, v_4 is adjacent to v_5, v_7 is adjacent to v_6 and v_8, and no other edges are present in $W''_n[\{v_1, \ldots, v_8\}]$. By Claim 4.40, $v_2 \in X \cup Y$ and at least one of v_1 and v_3 belongs to $X \cup Y$. Without loss of generality, $v_2 \in X$ and $v_1 \in X \cup Y$. As in the proof of Claim 4.41, $\{v_1, v_2\}$ dominates $X \cup Y$ and so $\{v_4, \ldots, v_8\} \in A \cup B \cup C$.

Suppose first that at least two vertices among v_6, v_7 and v_8 belong to the same partition class in $\{A, B, C\}$. These two vertices are non-adjacent, as A, B, C are independent sets, and so they must be v_6 and v_8. Without loss of generality, $v_6, v_8 \in R$ for some $R \in \{A, B, C\}$. Similarly, at least one of v_4 and v_5 does not belong to R, say $v_4 \in (A \cup B \cup C) \setminus R$. Then v_4 is non-adjacent to both $v_6, v_8 \in R$, contradicting Claim 4.39.

Therefore, v_6, v_7 and v_8 belong to distinct partition classes in $\{A, B, C\}$. By Claim 4.39, none of v_4 and v_5 belongs to the partition class of either v_6 or v_8. But then v_4 and v_5 both belong to the partition class of v_7, contradicting the fact that every class is an independent set. ◇

This concludes the proof of Theorem 4.36. □

4.4 Towards a dichotomy for the mim-width of (H_1, H_2)-free graphs

Combining the results of Sections 4.2.1 and 4.2.2, we can finally show Theorem 4.6, which we restate for convenience.

Theorem 4.6. *Let $r \geq 3$ and $s, t \geq 2$ be integers. Then the mim-width of the class of $(rP_1, \overline{K_{s,t} + P_1})$-free graphs is bounded if and only if:*

- *$r = 3$ and one of s and t is at most 3;*

- *$r = 4$ and one of s and t is at most 2.*

In all these cases, the mim-width is also quickly computable.

Proof. If $r \geq 5$, the mim-width is unbounded by Theorem 4.28. Suppose now that $r = 4$. If both s and t are at least 3, the mim-width is unbounded by Theorem 4.26, whereas if one of s and t is at most 2, the mim-width is bounded and quickly computable by Theorem 4.17. Finally, suppose that $r = 3$. If both s and t are at least 4, the mim-width is unbounded by Theorem 4.24, whereas if one of s and t is at most 3, the mim-width is bounded and quickly computable by Theorem 4.14. □

With the aid of Theorems 4.33 and 4.36, we can prove Theorem 4.10 and Theorem 4.9, which we restate for convenience.

Theorem 4.9. *Let $r \geq 5$ be an integer and let $H = sP_1 + tP_2 + uP_3$, for $s, t, u \geq 0$. Then exactly one of the following holds:*

- *$H \subseteq_i sP_1 + tP_2$, or $H \subseteq_i sP_1 + P_3$, and the mim-width of the class of (K_r, H)-free graphs is bounded and quickly computable;*

- *$H \supseteq_i P_3 + P_2 + P_1$, and the mim-width of the class of (K_r, H)-free graphs is unbounded;*

- *$H = 2P_3$, or $H = P_3 + P_2$.*

Proof. By Theorem 4.33, if H contains $P_3 + P_2 + P_1$, then the mim-width of the class of (K_r, H)-free graphs is unbounded. So we may assume that $u \leq 2$. If $u = 0$, then the mim-width is bounded by [26, Theorem 30-(xiv)]. If $u = 1$, then the mim-width is unbounded for $t \geq 2$ and $s \geq 0$ or $t = 1$ and $s \geq 1$ (Theorem 4.33), and bounded for $t = 0$ ([26, Theorem 30-(xii)]). This leaves open the case $H = P_3 + P_2$. Finally, if $u = 2$, then the mim-width is unbounded if one of t and s is at least 1. This leaves open the case $H = 2P_3$. □

Theorem 4.10. *Let $r = 4$ and let $H = sP_1 + tP_2 + uP_3$, for $s, t, u \geq 0$. Then exactly one of the following holds:*

- *$H \subseteq_i sP_1 + tP_2$, or $H \subseteq_i sP_1 + P_3$, and the mim-width of the class of (K_r, H)-free graphs is bounded and quickly computable;*
- *$H \supseteq_i P_3 + 2P_2 + P_1$, or $2P_3 + P_2$, and the mim-width of the class of (K_r, H)-free graphs is unbounded;*
- *$H = P_3 + 2P_2$, or $H = P_3 + P_2 + sP_1$, or $H = 2P_3 + sP_1$.*

Proof. By Theorem 4.36, if H contains $P_3 + 2P_2 + P_1$ or $2P_3 + P_2$, then the mim-width of the class of (K_r, H)-free graphs is unbounded. So we may assume that $u \leq 2$. If $u = 0$, then the mim-width is bounded by [26, Theorem 30-(xiv)]. If $u = 1$, then the mim-width is bounded for $t = 0$ ([26, Theorem 30-(xii)]), and unbounded for $t \geq 2$ and $s \geq 1$ or $t \geq 3$ and $s \geq 0$ (Theorem 4.36). This leaves open the cases $H = P_3 + 2P_2$ and $P_3 + P_2 + sP_1$. Finally, if $u = 2$, then the mim-width is unbounded for $t \geq 1$. This leaves open the case $H = 2P_3 + sP_1$. □

We leave the remaining cases as open problems:

Open Problem 1. *Determine the mim-width (un)boundedness of the class of $(K_r, 2P_3)$-free graphs and the class of $(K_r, P_3 + P_2)$-free graphs for $r \geq 5$*

Open Problem 2. *Determine the mim-width (un)boundedness of the class of $(K_4, P_3 + 2P_2)$-free graphs, the class of $(K_4, P_3 + P_2 + sP_1)$-free graphs and the class of $(K_4, 2P_3 + sP_1)$-free graphs for $s \geq 0$.*

Chapter 5

Sim-Width

This chapter contains joint work with Andrea Munaro: *On algorithmic applications of sim-width and mim-width of* (H_1, H_2)*-free graphs* [130] and joint work with Nick Brettell, Andrea Munaro and Daniël Paulusma: *Comparing Width Parameters on Graph Classes* [28].

5.1 Introduction

Sim-width, introduced in Kang et al. [111], is one of the most powerful graph width parameters that has been studied in the literature. The trade-off of working with a more powerful width parameter is that, typically, fewer problems admit a polynomial-time algorithm when the parameter is bounded. Consider, for example, mim-width and the more powerful sim-width. Dominating Set belongs to the framework of (σ, ρ)-vertex subset problem (it is a $(\mathbb{N} \cup \{0\}, \mathbb{N})$-vertex subset problem) and so XP parameterized by mim-width [32]. However, Dominating Set is NP-complete on chordal graphs, a class of graphs of sim-width 1 [111]. Therefore, unless P = NP, we can not expect Dominating Set to be polynomial-time solvable parameterized by sim-width. On the other hand, it is known that one can solve Independent Set and 3-Colouring in polynomial time on both chordal graphs and co-comparability graphs, two classes of sim-width at most 1, as shown by Kang et al. [111]. This led them to ask the following:

Question 3 (Kang et al. [111])**.** *Is any of* Independent Set *and* 3-Colouring NP*-complete on graphs of sim-width at most* 1*?*

To the best of our knowledge, no problem is known to be in XP parameterized by the sim-width of a given branch decomposition of the input graph. In Section 5.3 we show that for odd d, List (d, k)-Colouring is polynomial-time solvable for every graph class whose sim-width is bounded and quickly computable, thus answering in the negative one half of Question 3, unless $\mathsf{P} = \mathsf{NP}$

Theorem 5.1. *For every $k \geq 1$ and odd $d \geq 1$, List (d, k)-Colouring is in XP parameterized by the sim-width of a given branch decomposition of the input graph.*

Theorem 5.1 is proved by considering the natural question of whether boundedness of sim-width is preserved under graph powers. Note that this cannot hold for treewidth. However, Suchan and Todinca [144] showed that, for every positive integer r and graph G, $\text{nlcw}(G^r) \leq 2(r+1)^{\text{nlcw}(G)}$, where NLC-width (nlcw) is a parameter equivalent to clique-width. Jaffke et al. [106] showed that, for every positive integer r and graph G, $\text{mimw}(G^r) \leq 2\text{mimw}(G)$. Lima et al. [122] showed that, for every odd positive integer r and graph G, $\text{tree-}\alpha(G^r) \leq \text{tree-}\alpha(G)$ and that, for every fixed even positive integer r, there is no function f such that $\text{tree-}\alpha(G^r) \leq f(\text{tree-}\alpha(G))$ for all graphs G.

In Section 5.2, we show that sim-width behaves similarly to tree-independence number with respect to graph powers:

Theorem 5.2. *Let $r \geq 1$ be an odd integer and let G be a graph. If (T, δ) is a branch decomposition of G with $\text{simw}_G(T, \delta) = w$, then (T, δ) is also a branch decomposition of G^r with $\text{simw}_{G^r}(T, \delta) \leq w$. In particular, $\text{simw}(G^r) \leq \text{simw}(G)$, for every odd integer $r \geq 1$.*

Moreover, for every fixed even positive integer r, we observe that there is no function f such that $\text{simw}(G^r) \leq f(\text{simw}(G))$ for all graphs G. This helps to increase our understanding of the currently poorly understood width parameter sim-width.

In Section 5.4, we show that a negative answer to the other half of Question 3 would have important algorithmic implications for Max Weight Independent $\mathcal{H}$-Packing, a problem studied for example in [33, 48]. Before formulating it, we state some definitions and results. Let $\mathcal{H}$ be a set of connected graphs of G. The *$\mathcal{H}$-graph of G*, denoted $\mathcal{H}(G)$, is defined in [33] as follows: the vertex set is $\mathcal{H}$ and two distinct vertices are adjacent if and only if they either have a vertex in common or there is an edge in G connecting them. Cameron and Hell [33] showed that, for any set $\mathcal{H}$ of connected graphs, the $\mathcal{H}$-graph of any chordal graph is chordal. Dallard et al. [48] generalised this by showing that mapping any graph G to its $\mathcal{H}$-graph does not increase the tree-independence number. We show that this operation does not increase the sim-width either.

Theorem 5.3. *Let G be a graph and let (T, δ) be a branch decomposition of G. Let $\mathcal{H}$ be a non-empty finite set of connected non-null subgraphs of G and let r be the maximum number of vertices of a graph in $\mathcal{H}$. If $|V(\mathcal{H}(G))| > 1$, then we can obtain in $O(|V(G)|^{r+1})$ time a branch decomposition (T', δ') of $\mathcal{H}(G)$ such that $\mathrm{simw}_{\mathcal{H}(G)}(T', \delta') \leq \mathrm{simw}_G(T, \delta)$.*

Two subgraphs H_1 and H_2 of a graph G are *independent* if they are vertex-disjoint and no edge of G joins a vertex of H_1 with a vertex of H_2. An *independent $\mathcal{H}$-packing* in G is a set of pairwise independent subgraphs from $\mathcal{H}$. Given a graph G, a weight function $w\colon \mathcal{H} \to \mathbb{Q}_+$ on the subgraphs in $\mathcal{H}$, and an independent $\mathcal{H}$-packing P in G, the weight of P is defined as $\sum_{H \in P} w(H)$. Given a graph G and a weight function $w\colon \mathcal{H} \to \mathbb{Q}_+$, the Max Weight Independent $\mathcal{H}$-Packing problem asks to find an independent $\mathcal{H}$-packing in G of maximum weight. If all subgraphs in $\mathcal{H}$ have weight 1, we obtain the special case Max Independent $\mathcal{H}$-Packing. Max Weight Independent $\mathcal{H}$-Packing is a common generalisation of several problems studied in the literature, including Max Weight Independent Set, Max Weight Induced Matching, Dissociation Set and k-Separator (we refer to [48] for a comprehensive literature review).

Cameron and Hell [33] showed that Max Independent $\mathcal{H}$-packing is polynomial-time solvable, among others, for the following graph classes: weakly chordal graphs and hence chordal graphs, AT-free graphs and hence co-comparability graphs, circular-arc graphs, circle graphs. Dallard et al. [48] showed that Max Weight Independent $\mathcal{H}$-Packing is polynomial-time solvable for every graph class whose tree-independence number is bounded and quickly computable.

Theorem 5.4. *Let $\mathcal{H}$ be a non-empty finite set of connected non-null graphs such that each graph in $\mathcal{H}$ has at most r vertices. Let $\mathcal{G}$ be a graph class whose sim-width is bounded and quickly computable. If* Max Weight Independent Set *is polynomial-time solvable for $\mathcal{G}$, then* Max Weight Independent $\mathcal{H}$-Packing *is polynomial-time solvable for $\mathcal{G}$. Similarly, if* Independent Set *is polynomial-time solvable for $\mathcal{G}$, then* Max Independent $\mathcal{H}$-Packing *is polynomial-time solvable for $\mathcal{G}$.*

In Section 5.5 and Section 5.6, we focus on line graphs. First we prove Theorem 5.5, which states that the sim-width of a line graph does not increase after edge contractions of the original graph. This theorem will be used to show Theorem 6.6 in Section 6.4.1, which states that a graph class has bounded treewidth if and only if the class of their line graphs has bounded sim-width.

Theorem 5.5. *Let G be a graph with $|E(G)| \geq 3$ and let G' be the graph obtained by contracting an edge of G. Then* $\mathrm{simw}(L(G)) \geq \mathrm{simw}(L(G'))$.

We then give the exact value of $\mathrm{simw}(L(K_{n,m}))$:

Theorem 5.6. *Let n and m be integers with $6 < n \leq m$. Then* $\mathrm{simw}(L(K_{n,m})) = \lceil \frac{n}{3} \rceil$.

Note that this result implies that $\lceil \frac{n}{6} \rceil \leq \mathrm{simw}(L(K_n)) \leq \lceil \frac{2n}{3} \rceil$, though we are unable to determine its exact value.

5.2 Sim-width of graph powers

In this section we prove Theorem 5.2 and observe that a result similar to Theorem 5.2 cannot hold for even powers. This result has important consequences for List (d, k)-Colouring.

Theorem 5.2. *Let $r \geq 1$ be an odd integer and let G be a graph. If (T, δ) is a branch decomposition of G with* $\mathrm{simw}_G(T, \delta) = w$, *then (T, δ) is also a branch decomposition of G^r with* $\mathrm{simw}_{G^r}(T, \delta) \leq w$. *In particular,* $\mathrm{simw}(G^r) \leq \mathrm{simw}(G)$, *for every odd integer $r \geq 1$.*

Proof. Clearly, (T, δ) is a branch decomposition of G^r. Suppose, to the contrary, that $\mathrm{simw}_{G^r}(T, \delta) > w$. Hence, $r \geq 3$. Let $e \in E(T)$ be such that $\mathrm{cutsim}_{G^r}(A_e, \overline{A_e}) \geq w + 1$. There exist independent sets $X = \{x_1, \dots, x_{w+1}\} \subseteq A_e$ and $Y = \{y_1, \dots, y_{w+1}\} \subseteq \overline{A_e}$ of G^r such that, for each $i, j \in \{1, \dots, w + 1\}$, x_i is adjacent to y_j if and only if $i = j$. Since x_i is adjacent to y_i in G^r for each $i \in \{1, \dots, w + 1\}$, there exists a path $P_i = v(i, 0)v(i, 1)v(i, 2) \cdots v(i, a_i)$ in G, with endpoints $v(i, 0) = x_i$ and $v(i, a_i) = y_i$, for some $a_i \leq r$. We claim that, for each $i \neq j$, $V(P_i) \cap V(P_j) = \varnothing$ and no vertex of P_i is adjacent to a vertex of P_j in G. Suppose, to the contrary, that there exist $v(i, k) \in V(P_i)$ adjacent to $v(j, l) \in V(P_j)$ (the proof that $V(P_i) \cap V(P_j) = \varnothing$ is similar and left to the reader).

Suppose first that $k+l < r$. Then, there is a walk $v(i,0)v(i,1)\cdots v(i,k)v(j,l)v(j,l-1)\cdots v(j,0)$ in G of length at most r and hence a $v(i,0), v(j,0)$-path in G, from which $x_i = v(i,0)$ is adjacent to $x_j = v(j,0)$ in G^r, a contradiction. Suppose now that $k+l > r$. Then, there is a walk $v(i,a_i)v(i,a_i-1)\cdots v(i,k)v(j,l)v(j,l+1)\cdots v(j,a_j)$ in G of length at most r, from which $y_i = v(i,a_i)$ is adjacent to $y_j = v(j,a_j)$ in G^r, a contradiction. Suppose finally that $k+l = r$. Since r is odd, either $k < l$ or $k > l$. Without loss of generality, $k < l$. Then, there is a walk $v(i,0)v(i,1)\cdots v(i,k)v(j,l)v(j,l+1)\cdots v(j,a_j)$ in G of length at most r, and so $x_i = v(i,0)$ is adjacent to $y_j = v(j,a_j)$ in G^r, a contradiction.

Consider now, for each $i \in \{1,\ldots,w+1\}$, the path $P_i = v(i,0)v(i,1)v(i,2)\cdots v(i,a_i)$. Since $x_i = v(i,0) \in A_e$ and $y_i = v(i,a_i) \in \overline{A_e}$, there exists an integer $b_i \leq a_i - 1$ such that $v(i,b_i) \in A_e$ and $v(i,b_i+1) \in \overline{A_e}$. Let $x'_i = v(i,b_i)$ and $y'_i = v(i,b_i+1)$. Clearly, each x'_i is distinct from and adjacent to y'_i. Moreover, by the paragraph above, each x'_i is distinct from and non-adjacent to both x'_j and y'_j, for $i \neq j$. We then have that $X' = \{x'_1,\ldots,x'_{w+1}\} \subseteq A_e$ and $Y' = \{y'_1,\ldots,y'_{w+1}\} \subseteq \overline{A_e}$ are independent sets of G and $G[X',Y'] \cong (w+1)P_2$. Therefore, $\text{cutsim}_G(A_e,\overline{A_e}) \geq w+1$, a contradiction. □

Lima et al. [122] showed that, for every fixed even integer $r \geq 2$ and for every graph H, there exists a chordal graph G such that G^r contains an induced subgraph isomorphic to H. Since walls have arbitrarily large sim-width, their result immediately implies the following:

Lemma 5.7. *For every even integer $r \geq 2$ and every integer $w \geq 1$, there exists a graph G such that* $\text{simw}(G) = 1$ *while* $\text{simw}(G^r) \geq w$. *In particular, for every fixed even integer $r \geq 2$, there is no function f such that* $\text{simw}(G^r) \leq f(\text{simw}(G))$ *for all graphs G.*

5.3 Sim-width and Colouring

We begin this section with a result relating sim-width and mim-width but first require a series of definitions. Let $K_t \boxminus K_t$ be the graph obtained from $2K_t$ by adding a perfect matching and let $K_t \boxminus S_t$ be the graph obtained from $K_t \boxminus K_t$ by removing all the edges in one of the complete graphs. Recall that $R(s,t)$ denotes the minimum number such that any graph on at least $R(s,t)$ vertices contains either a clique of size s, or an independent set of size t.

Proposition 5.8 (Kang et al. [111]). *Let G be a graph with* $\mathrm{simw}(G) = w$ *and no induced subgraph isomorphic to* $K_t \boxminus K_t$ *and* $K_t \boxminus S_t$. *Then* $\mathrm{mimw}(G) \leq R(R(w+1,t), R(t,t))$.

Essentially, the proposition above states that sim-width is equivalent to mim-width in graph classes that do not contain large cliques. This immediately implies the following:

Corollary 5.9. *For every $k \geq 1$,* List k-Colouring *is polynomial-time solvable for every graph class whose sim-width is bounded and quickly computable.*

Proof. Given an instance consisting of a graph G and a k-list assignment L, together with a branch decomposition (T, δ) of G with $\mathrm{simw}_G(T, \delta) = w$, we proceed as follows. We check in polynomial time whether G contains a copy of K_{k+1}. If it does, then we have a no-instance. Otherwise, G is K_{k+1}-free. Then, by Propoistion 5.8, (T, δ) has mim-width at most $R(R(w + 1, k+1), R(k+1, k+1))$, and we simply apply Theorem 4.7. This concludes the proof. □

It is worth noticing that Corollary 5.9 does not really give wider applicability when compared to Theorem 4.7. Indeed, input graphs of List k-Colouring can always be assumed to be K_{k+1}-free and every subclass of K_{k+1}-free graphs has bounded sim-width if and only if it has bounded mim-width: This follows from Proposition 5.8 and the fact that $\mathrm{simw}(G) \leq \mathrm{mimw}(G)$ for any graph G. However, using Theorem 5.2 we can obtain a non-trivial result for List (d, k)-Colouring.

A (d, k)-*colouring* of a graph G is an assignment of colours to the vertices of G using at most k colours such that no two vertices at distance at most d receive the same colour. For fixed d and k, (d, k)-Colouring is the problem of determining whether a given graph G has a (d, k)-colouring. The List (d, k)-Colouring problem requires in addition that every vertex u must receive a colour from some given set $L(u) \subseteq \{1, \dots, k\}$. Sharp [142] provided the following complexity dichotomy: For fixed $d \geq 2$, (d, k)-Colouring is polynomial-time solvable for $k \leq \lfloor \frac{3d}{2} \rfloor$ and NP-hard for $k > \lfloor \frac{3d}{2} \rfloor$.

Clearly, a graph G has a (d, k)-colouring if and only if G^d has a $(1, k)$-colouring. Hence, Theorem 5.2 immediately implies the following:

Theorem 5.1. *For every $k \geq 1$ and odd $d \geq 1$,* List (d, k)-Colouring *is in* XP *parameterized by the sim-width of a given branch decomposition of the input graph.*

Kratsch and Müller [115] showed that List $(1, k)$-Colouring is polynomial-time solvable for AT-free graphs and hence for the subclass of cocomparability graphs. Moreover, Chang et al. [38] showed that if G is an AT-free graph, then G^d is a cocomparability graph for any $d \geq 2$ (see also [25]). Therefore, for any $k, d \geq 1$, List (d, k)-Colouring is polynomial-time solvable for AT-free graphs. Since we can compute in linear time a branch decomposition of a cocomparability graph with sim-width at most 1 [111], Corollary 5.1 implies the following special case:

Corollary 5.10. *For every $k \geq 1$ and odd $d \geq 1$,* List (d, k)-Colouring *is polynomial-time solvable for AT-free graphs.*

5.4 Sim-width and Maximum Weight Independent Packing

In this section we prove Theorem 5.3 and Theorem 5.4.

Theorem 5.3. *Let G be a graph and let (T, δ) be a branch decomposition of G. Let $\mathcal{H}$ be a non-empty finite set of connected non-null subgraphs of G and let r be the maximum number of vertices of a graph in $\mathcal{H}$. If $|V(\mathcal{H}(G))| > 1$, then we can obtain in $O(|V(G)|^{r+1})$ time a branch decomposition (T', δ') of $\mathcal{H}(G)$ such that $\mathrm{simw}_{\mathcal{H}(G)}(T', \delta') \leq \mathrm{simw}_G(T, \delta)$.*

Proof. Observe that if G is edgeless, then $\mathcal{H}(G)$ is edgeless as well and the statement trivially holds. Therefore, we assume that G is not edgeless, and hence $\mathrm{simw}_G(T, \delta) \geq 1$.

Let $\mathcal{H} = \{H_1, \ldots, H_n\}$. Let h be an arbitrary vertex of $\mathcal{H}(G)$. Hence, h corresponds to a subgraph H_i of G, for some $i \in \{1, \ldots, n\}$. We now arbitrarily order $V(G)$ and let $f(h)$ be the smallest vertex in h with respect to this ordering. For $v \in V(G)$, let $F(v) = \{h \in V(\mathcal{H}(G)) : f(h) = v\}$. Note that $F(v)$ is a clique in $\mathcal{H}(G)$. We can compute all sets $F(v)$, for $v \in V(G)$, in $O(|V(G)| \cdot |V(\mathcal{H}(G))|) = O(|V(G)|^{r+1})$ time.

We are now ready to construct (T', δ') from (T, δ) as follows (see Figure 5.1). For each leaf $t \in V(T)$, we let $v_t = \delta^{-1}(t)$, and do the following. If $F(v_t) \neq \varnothing$, we distinguish two cases. Suppose first that $|F(v_t)| = 1$. In this case, build a 1-caterpillar C_t and add the edge connecting the single vertex x_t in the backbone of C_t with the node t. Suppose now that $|F(v_t)| \geq 2$. In this case, build a $|F(v_t)|$-caterpillar C_t, subdivide an arbitrary edge of the backbone of C_t by adding a new vertex x_t and add the edge $x_t t$. Finally, if $F(v_t) = \varnothing$, trim the leaf t of T. Observe that, since $|V(\mathcal{H}(G))| > 1$, either there exists a leaf $t \in V(T)$ such that $|F(v_t)| \geq 2$ or there exist at least two leaves $t_1, t_2 \in V(T)$ such that $|F(v_{t_1})| \geq 1$ and $|F(v_{t_2})| \geq 1$. This implies that each leaf t of T such that $F(v_t) = \varnothing$ can be trimmed. Moreover, by definition, no new leaf is created after an application of trimming. Let T' be the tree obtained by the procedure above. Let δ' be the map from $V(\mathcal{H}(G))$ to the leaves of T' which restricted to $F(v_t)$ is an arbitrary bijection from $F(v_t)$ to the leaves of C_t for each t. It is easy to see that (T', δ') is a branch decomposition of $\mathcal{H}(G)$ and that it can be computed in $O(|V(G)|^2)$ time.

We now show that $\text{simw}_{\mathcal{H}(G)}(T', \delta') \leq \text{simw}_G(T, \delta)$. Suppose that $\text{simw}_{\mathcal{H}(G)}(T', \delta') = k$. Since the statement is trivially true if $k \leq 1$, we may assume $k \geq 2$. Each $e' \in E(T')$ naturally induces a partition $(A_{e'}, \overline{A_{e'}})$ of $V(\mathcal{H}(G))$. Consider $e \in E(T')$ such that $\text{cutsim}_{\mathcal{H}(G)}(A_e, \overline{A_e}) = \text{simw}_{\mathcal{H}(G)}(T', \delta') = k$. Then, there is a matching $\{x_1'y_1', \ldots, x_k'y_k'\}$ of size k such that $\{x_1', \ldots, x_k'\} \subseteq A_e$ and $\{y_1', \ldots, y_k'\} \subseteq \overline{A_e}$ are independent sets of $\mathcal{H}(G)$. Suppose first that e is an edge of C_t or the edge $x_t t$, for some leaf $t \in V(T)$. Then, one of A_e and $\overline{A_e}$ is a subset of $F(v_t)$, where $v_t = \delta^{-1}(t)$. Since each $F(v_t)$ is a clique in $\mathcal{H}(G)$, we have that $k \leq 1$. Hence, we may assume that $e \in E(T') \cap E(T)$. Then, for any $h \in V(\mathcal{H}(G))$, $\delta'(h)$ and $\delta(f(h))$ belong to the same component of $T' - e$, and so e naturally induces a partition $(A_e, \overline{A_e})$ of $V(\mathcal{H}(G))$ and a partition $(B_e, \overline{B_e})$ of $V(G)$ satisfying the following property: For any $h \in V(\mathcal{H}(G))$, $h \in A_e$ if and only if $f(h) \in B_e$.

For $v \in V(\mathcal{H}(G))$, let $H(v)$ be its corresponding subgraph in G. We claim that, for $i \neq j$, $H(x_i') + H(y_i')$ and $H(x_j') + H(y_j')$ are disjoint and anticomplete in G. Indeed, suppose without loss of generality that $H(x_i')$ shares a vertex with either $H(x_j')$ or $H(y_j')$. Then, x_i' is adjacent to either x_j' or y_j' in $\mathcal{H}(G)$, a contradiction. Similarly, if there is an edge between $H(x_i')$ and either $H(x_j')$ or $H(y_j')$ in G, then x_i' is adjacent to either x_j' or y_j' in $\mathcal{H}(G)$, a contradiction again.

We now claim that $G[H(x_i') + H(y_i')]$ is connected. Clearly $H(x_i')$ and $H(y_i')$ are connected. Moreover, since x_i' is adjacent to y_i', either $H(x_i')$ shares a vertex with $H(y_i')$ or there is an edge in G between $H(x_i')$ and $H(y_i')$. In either case we obtain that $G[H(x_i') = H(y_i')]$ is connected.

Therefore, for each $i \in \{1, \ldots, k\}$, there is a path P_i in $G[H(x_i') + H(y_i')]$ from $f(x_i')$ to $f(y_i')$ in G, say $P_i = v_0 v_1 \cdots v_\ell$ where $v_0 = f(x_i')$ and $v_\ell = f(y_i')$. Since $x_i' \in A_e$ and $y_i' \in \overline{A_e}$, it follows that $f(x_i') \in B_e$ and $f(y_i') \in \overline{B_e}$. Since the path P_i must cross the cut $(B_e, \overline{B_e})$ of G, there exists $q \in \{0, \ldots, \ell - 1\}$ such that $v_q \in B_e$ and $v_{q+1} \in \overline{B_e}$. We let $x_i = v_q$ and

$y_i = v_{q+1}$. Clearly, $x_i y_i \in E(G)$. We now claim that, for each $i \neq j$, $\{x_i, y_i\}$ and $\{x_j, y_j\}$ are disjoint and anticomplete in G. This simply follows from the fact that, for $p \in \{i, j\}$, $\{x_p, y_p\} \subseteq G[H(x'_p) + H(y'_p)]$ and $G[H(x'_i) + H(y'_i)]$ and $G[H(x'_j) + H(y'_j)]$ are disjoint and anticomplete in G.

Let now $X = \{x_1, \ldots, x_k\}$ and $Y = \{y_1, \ldots, y_k\}$. By the previous paragraph, $X \subseteq B_e$ and $Y \subseteq \overline{B_e}$, X and Y are independent sets and $G[X, Y] \cong kP_2$. Therefore, $\mathrm{simw}_G(T, \delta) \geq \mathrm{cutsim}_G(B_e, \overline{B_e}) \geq k = \mathrm{simw}_{\mathcal{H}(G)}(T', \delta')$. □

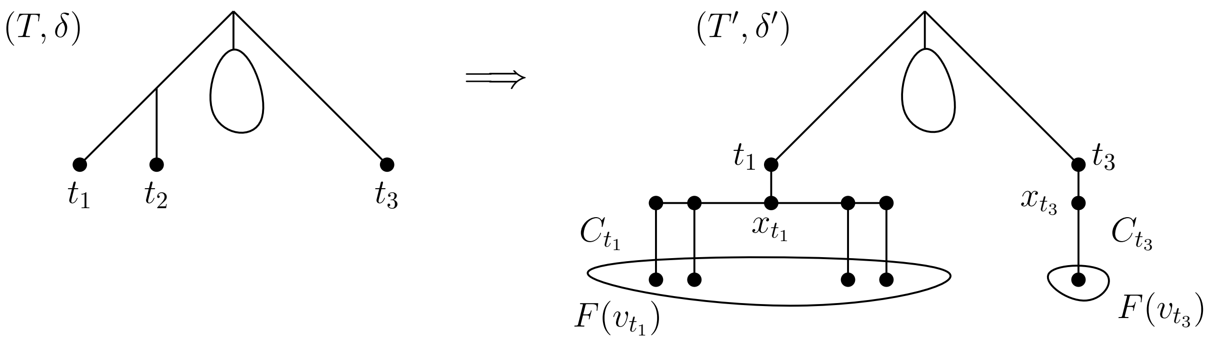

Figure 5.1: How to construct a branch decomposition (T', δ') of $\mathcal{H}(G)$ from a branch decomposition (T, δ) of G. We distinguish vertices t_i such that $|F(v_{t_i})| = 0$ $(i = 2)$, $|F(v_{t_i})| = 1$ $(i = 3)$ and $|F(v_{t_i})| \geq 2$ $(i = 1)$.

Besides Theorem 5.3, in order to show Theorem 5.4, we need the following two results.

Theorem 5.11 (Dallard et al. [48])**.** *Let $\mathcal{H}$ be a non-empty finite set of connected non-null graphs and let r be the maximum number of vertices of a graph in $\mathcal{H}$. Then there exists an algorithm that takes as input a graph G and computes the graph $\mathcal{H}(G)$ in $O(|V(G)|^{2r})$ time.*

Observation 5.12 (Dallard et al. [48])**.** *Let $\mathcal{H}$ be a finite set of connected non-null graphs. Let G be a graph and let $w\colon \mathcal{H} \to \mathbb{Q}_+$. Let I be an independent set in $\mathcal{H}(G)$ of maximum weight with respect to the weight function w. Then I is an independent $\mathcal{H}$-packing in G of maximum weight.*

Theorem 5.4. *Let $\mathcal{H}$ be a non-empty finite set of connected non-null graphs such that each graph in $\mathcal{H}$ has at most r vertices. Let $\mathcal{G}$ be a graph class whose sim-width is bounded and quickly computable. If* Max Weight Independent Set *is polynomial-time solvable for $\mathcal{G}$, then* Max Weight Independent $\mathcal{H}$-Packing *is polynomial-time solvable for $\mathcal{G}$. Similarly, if* Independent Set *is polynomial-time solvable for $\mathcal{G}$, then* Max Independent $\mathcal{H}$-Packing *is polynomial-time solvable for $\mathcal{G}$.*

Proof. Given the input graph $G \in \mathcal{G}$, we compute in polynomial time a branch decomposition of G of sim-width at most k, for some integer k. We then compute $\mathcal{H}(G)$ in polynomial time using Theorem 5.11. If $|V(\mathcal{H}(G))| \leq 1$, we immediately conclude thanks to Observation 5.12. Otherwise, by Theorem 5.3, we compute in polynomial time a branch decomposition of $\mathcal{H}(G)$ of sim-width at most k. We finally compute in polynomial time a maximum-weight independent set in $\mathcal{H}(G)$ which, by Observation 5.12, is an independent $\mathcal{H}$-packing in G of maximum weight. □

Remark 5.13. Using Theorem 5.2 we can extend Theorem 5.4 to Max Weight Distance-d Independent $\mathcal{H}$-Packing (the problem of finding a set of subgraphs that are pairwise of distance at least d of maximum weight) for even d. However, it requires a different formulation Theorem 5.3, hence omitted here.

5.5 Sim-width of line graphs and edge contractions

In this section we start working with line graphs and prove Theorem 5.5.

Theorem 5.5. *Let G be a graph with $|E(G)| \geq 3$ and let G' be the graph obtained by contracting an edge of G. Then* $\mathrm{simw}(L(G)) \geq \mathrm{simw}(L(G'))$.

Proof. Suppose that G' is obtained from G by contracting the edge uv into the vertex w. Let $X = \{x_1, \ldots, x_n\}$ be the set of vertices in G adjacent to u but not v. Let $Y = \{y_1, \ldots, y_m\}$ be the set of vertices in G adjacent to v but not u. Let $Z = \{z_1, \ldots, z_\ell\}$ be the set of vertices in G adjacent to both u and v. Suppose that $\mathrm{simw}(L(G)) = k$. Then, there exists a branch decomposition (T, δ) of $L(G)$ such that $\mathrm{simw}_{L(G)}(T, \delta) = k$. We show how to construct a branch decomposition (T', δ') of $L(G')$ such that $\mathrm{simw}_{L(G')}(T', \delta') \leq k$. This would imply that $\mathrm{simw}(L(G')) \leq \mathrm{simw}_{L(G')}(T', \delta') \leq k = \mathrm{simw}(L(G))$, thus concluding the proof. We may assume that $L(G') \ncong K_1$, for otherwise $\mathrm{simw}(L(G')) = 0$ and $\mathrm{simw}(L(G)) \geq 0$.

We construct (T', δ') from (T, δ) as follows. First, we trim the leaf $\delta(uv)$ of T. Then, for each z_i, we recursively trim the leaf $\delta(z_i v)$. We call the resulting tree T'. We now argue that these operations can indeed be performed. Since we aim to trim $\ell + 1$ leaves, it is enough to show that T contains at least $\ell + 1$ vertices of degree at least 3. We first recursively contract edges of T having at least one endpoint of degree 2. The resulting tree $\widetilde{T}$ has each of its nodes of degree either 1 or 3. Moreover, the number of degree-3 vertices in $\widetilde{T}$ is the same as that of T and the same holds for degree-1 vertices. Let d_1 and d_3 be the number of degree-1 and degree-3 vertices in $\widetilde{T}$ (and hence in T). It is easy to see that $d_1 - d_3 = 2$. Since the leaves of T are in bijection with the edges of G, we obtain that $d_3 = |E(G)| - 2$. Thus, it suffices to show that $|E(G)| \geq \ell + 3$. We now count the edges of G. Since $uv \in E(G)$ and, for each $i \in \{1, \dots, \ell\}$, we have $\{z_i u, z_i v\} \subseteq E(G)$, we see that $|E(G)| = 1 + 2\ell + k$, for some $k \geq 0$. In particular, as $|E(G)| \geq 3$, either $|E(G)| \geq \ell + 3$, or $k = 0$ and $\ell = 1$. But in the latter case $L(G') \cong K_1$. Therefore T' is a well-defined subcubic tree. To conclude the construction of (T', δ'), we define δ' as follows. For each x_i, let $\delta'(x_i w) = \delta(x_i u)$; for each y_i, let $\delta'(y_i w) = \delta(y_i v)$; for each z_i, let $\delta'(z_i w) = \delta(z_i u)$; finally, let δ' coincide with δ on the remaining vertices of $L(G')$ (these correspond to the edges of G not adjacent to uv).

We now show that $\mathrm{simw}_{L(G')}(T', \delta') \leq k$. Suppose, to the contrary, that $\mathrm{simw}_{L(G')}(T', \delta') \geq k+1$. Then, there exists $e \in E(T')$ such that $\mathrm{cutsim}_{L(G')}(A'_e, \overline{A'_e}) \geq k+1$. Since T' is obtained from T by repeated applications of trimming, and hence by repeated vertex deletions, $e \in E(T') \cap E(T)$. Then, e naturally induces a partition $(A_e, \overline{A_e})$ of $V(L(G))$. Without loss of generality, we assume that A_e agrees with A'_e on the vertices of $L(G)$ corresponding to edges of G not intersecting $\{u, v\}$, and the same for $\overline{A_e}$ and $\overline{A'_e}$. Now, since $\mathrm{cutsim}_{L(G')}(A'_e, \overline{A'_e}) \geq k+1$, there exist independent sets $P' = \{p_1, \dots, p_{k+1}\} \subseteq A'_e$ and $Q' = \{q_1, \dots, q_{k+1}\} \subseteq \overline{A'_e}$ of $L(G')$ such that $L(G')[P', Q'] \cong (k+1)P_2$. If none of the vertices in $P' \cup Q'$ correspond to an edge of G' incident to w, then $P' \subseteq A_e$ and $Q' \subseteq \overline{A_e}$, and P' and Q' are independent sets of $L(G)$ such that $L(G)[P', Q'] \cong (k+1)P_2$. This implies that $\mathrm{simw}_{L(G)}(T, \delta) \geq k+1$, a contradiction. Hence, there is a vertex in $P' \cup Q'$, say without loss of generality p_1, which corresponds to an edge of G' incident to w. Moreover, since p_1 is anticomplete to $\{p_j, q_j\}$ in $L(G')$, for each $j \geq 2$, the following claim holds:

Claim 5.14. *For each $j \geq 2$, neither of the edges p_j and q_j of G' is adjacent to p_1.*

We now introduce the following notation. If $p_1 = x_i w$, we let $a = x_i$ and $b = u$. If $p_1 = y_i w$, we let $a = y_i$ and $b = v$. If $p_1 = z_i w$, we let $a = z_i$ and $b = u$. Note that, in each case, ab is an edge of G and hence a vertex of $L(G)$.

Claim 5.15. *Let $P = \{ab, p_2, \dots, p_{k+1}\}$. Then $P \subseteq A_e$ and P is an independent set of $L(G)$.*

Proof of Claim 5.15. By Claim 5.14, for each $j \geq 2$, the edge p_j of G' is incident to neither w nor a. Moreover, $b \in V(G) \setminus V(G')$. These two facts imply that $P = \{ab, p_2, \ldots, p_{k+1}\} \subseteq E(G)$ and that P is an independent set of $L(G)$. By construction, $\delta'(p_1) = \delta'(aw) = \delta(ab)$, from which $ab \in A_e$ and so $P \subseteq A_e$. $\diamond$

To complete the proof of Theorem 5.5, we distinguish two cases.

Case 1: The edge q_1 of G' is not incident to w.

By Claim 5.15, $P = \{ab, p_2, \ldots, p_{k+1}\} \subseteq A_e$ and P is an independent set of $L(G)$. Since q_1 is not incident to w, it must be that $q_1 \in E(G)$. Similarly, by Claim 5.14, no q_j with $j \geq 2$ is incident to w and so $Q' = \{q_1, \ldots, q_{k+1}\} \subseteq E(G)$. Therefore, $Q' \subseteq \overline{A_e}$ and Q' is an independent set of $L(G)$. We now show that, in $L(G)$, the vertex ab is adjacent to the vertex q_1. As the vertex p_1 is adjacent to the vertex q_1 in $L(G')$, the edge q_1 is adjacent to the edge $p_1 = aw$. But by assumption, the edge q_1 is not incident to w, and so q_1 is incident to a. Hence, in G, the edge q_1 is adjacent to the edge ab (note that, since $q_1 \in E(G') \cap E(G)$, no endpoint of q_1 belongs to $\{u, v\}$) and so, in $L(G)$, the vertex q_1 is adjacent to the vertex ab. Observe finally that, since $Q' \subseteq V(L(G'))$, none of the edges of G' in Q' are incident to the vertex $b \in V(G) \setminus V(G')$. Combining this with Claim 5.14 and the fact that ab is adjacent to q_1 in $L(G)$, we obtain that $L(G)[P, Q'] \cong (k+1)P_2$. Therefore, $\mathrm{cutsim}_{L(G)}(A_e, \overline{A_e}) \geq k+1$ and so $\mathrm{simw}_{L(G)}(T, \delta) \geq k+1$, a contradiction.

Case 2: The edge q_1 of $E(G')$ is incident to w.

Consider $uv \in V(L(G))$. Either $uv \in A_e$ or $uv \in \overline{A_e}$.

Case 2.1: $uv \in \overline{A_e}$.

By Claim 5.15, $P = \{ab, p_2, \ldots, p_{k+1}\} \subseteq A_e$ is an independent set of $L(G)$. By assumption and Claim 5.14, $Q = \{uv, q_2, \ldots, q_{k+1}\} \subseteq \overline{A_e}$. Moreover, again by Claim 5.14, none of the edges q_j with $j \geq 2$ is adjacent to uv in G, or else they would be adjacent to p_1 in G'. Therefore, Q is an independent set of $L(G)$. It is finally easy to see that, in $L(G)$, ab is adjacent to uv, no q_j with $j \geq 2$ is adjacent to ab, and no p_j with $j \geq 2$ is adjacent to uv. Hence, $L(G)[P, Q] \cong (k+1)P_2$. Therefore, $\mathrm{cutsim}_{L(G)}(A_e, \overline{A_e}) \geq k+1$ and so $\mathrm{simw}_{L(G)}(T, \delta) \geq k+1$, a contradiction.

Case 2.2: $uv \in A_e$.

Let $q_1 = cw$, for some $c \in X \cup Y \cup Z$. By construction, $\delta'(q_1) = \delta'(cw) = \delta(cd)$, for some $d \in \{u, v\}$. Since $q_1 \in \overline{A'_e}$, we have that $cd \in \overline{A_e}$. Let $P = \{uv, p_2, \dots, p_{k+1}\}$ and $Q = \{cd, q_2, \dots, q_{k+1}\}$. By symmetry, the same proof as in Case 2.1 applies to show that $P \subseteq A_e$ and $Q \subseteq \overline{A_e}$ are independent sets in $L(G)$ such that $L(G)[P, Q] \cong (k+1)P_2$. Therefore, $\mathrm{cutsim}_{L(G)}(A_e, \overline{A_e}) \geq k+1$ and so $\mathrm{simw}_{L(G)}(T, \delta) \geq k+1$, a contradiction. □

5.6 Sim-width of $L(K_{n,m})$ and $L(K_n)$

In this section, we determine the exact value of $\mathrm{simw}(L(K_{n,m}))$. For a positive integer k, we use the notation $[k] = \{1, \dots, k\}$. For positive integers n and m, the $n \times m$ *rook graph* $R_{n,m}$ is the graph representing the legal moves of a rook on an $n \times m$ chessboard: the vertex set is $[n] \times [m]$, two vertices being adjacent if and only if they have one of the two coordinates in common. Clearly, $R_{n,m}$ is isomorphic to $K_n \square K_m$, the Cartesian product of K_n and K_m.

Observation 5.16 (Folklore). *$L(K_{n,m})$ is isomorphic to $R_{n,m}$.*

Theorem 5.6. *Let n and m be integers with $6 < n \leq m$. Then* $\mathrm{simw}(L(K_{n,m})) = \lceil \frac{n}{3} \rceil$.

Proof. In view of Observation 5.16, we show that $\mathrm{simw}(R_{n,m}) = \lceil n/3 \rceil$. We first need the following definition. For an integer $l \geq 1$, an *l-caterpillar* is a subcubic tree T on $2l$ vertices with $V(T) = \{s_1, \dots, s_l, t_1, \dots, t_l\}$, such that $E(T) = \{s_i t_i : 1 \leq i \leq l\} \cup \{s_i s_{i+1} : 1 \leq i \leq l-1\}$. Note that we label the leaves of an l-caterpillar $t_1, t_2, \dots, t_l$, in this order.

Let us begin by showing the upper bound $\mathrm{simw}(R_{n,m}) \leq \lceil n/3 \rceil$. Let $a = \lceil n/3 \rceil$ and $b = \lfloor 2n/3 \rfloor$. Observe that $1 \leq a < b < n$. Build an $(a+1)$-caterpillar C_1 with leaves $p_1, l_1, l_2, \dots, l_a$, a $(b-a+1)$-caterpillar C_2 with leaves $p_2, l_{a+1}, l_2, \dots, l_b$ and an $(n-b+1)$-caterpillar C_3 with leaves $p_3, l_{b+1}, l_2, \dots, l_n$. For each $i \in [n]$, build an $(m+1)$-caterpillar D_i with leaves $h_i, l_{i,1}, \dots, l_{i,m}$ and add the edge $h_i l_i$. Finally, add a vertex p_0 and add edges $p_0 p_1, p_0 p_2$ and $p_0 p_3$. Let T be the resulting tree and let δ be the function mapping each $(i, j) \in [n] \times [m]$ to the leaf $l_{i,j}$ of T. Clearly, (T, δ) is a branch decomposition of $R_{n,m}$. We now show that $\mathrm{simw}_{R_{n,m}}(T, \delta) \leq \lceil n/3 \rceil$.

Let $e \in E(T)$ and let $(A_e, \overline{A_e})$ be the corresponding bipartition of $V(R_{n,m})$. Suppose first that $e = h_i l_i$ or $e \in E(D_i)$, for some $i \in [n]$. Then, without loss of generality, the first coordinate of each vertex in A_e equals i and so A_e is a clique in $R_{n,m}$. Therefore, $\text{cutsim}_{R_{n,m}}(A_e, \overline{A_e}) \leq 1$. Suppose instead that $e = p_0p_j$ or $e \in E(C_j)$, for some $j \in \{1, 2, 3\}$. Then, without loss of generality, each vertex in A_e has first coordinate between 1 and $a = \lceil n/3 \rceil$, if $j = 1$, or between $a + 1 = \lceil n/3 \rceil + 1$ and $b = \lfloor 2n/3 \rfloor$, if $j = 2$, or between $b + 1 = \lfloor 2n/3 \rfloor + 1$ and n, if $j = 3$. In any case, it is easy to see that there are at most $\lceil n/3 \rceil$ distinct choices for the first coordinate, and so $\alpha(R_{n,m}[A_e]) \leq \lceil n/3 \rceil$, from which $\text{cutsim}_{R_{n,m}}(A_e, \overline{A_e}) \leq \lceil n/3 \rceil$.

Finally, we show that $\text{simw}(R_{n,m}) \geq \lceil n/3 \rceil$. Let $D = \{(i, i) : 1 \leq i \leq n\}$. Suppose, to the contrary, that $\text{simw}(R_{n,m}) < \lceil n/3 \rceil$ and let (T, δ) be a branch decomposition with sim-width $w < \lceil n/3 \rceil$. We first show that there exists $e \in E(T)$ such $|D|/3 \leq |A_e \cap D|, |\overline{A_e} \cap D|$. Indeed, by trimming the set of leaves $\delta(V(R_{n,m}) \setminus D)$ of T, we obtain a branch decomposition of $R_{n,m}[D]$ with sim-width at most w. We then apply [111, Lemma 2.3][1] to the graph $R_{n,m}[D]$ and the obtained branch decomposition. Fix now e as above and suppose that $A_e \cap D = \{(a_1, a_1), \dots, (a_r, a_r)\}$ and $\overline{A_e} \cap D = \{(b_1, b_1), \dots, (b_s, b_s)\}$, for pairwise distinct $a_1, \dots, a_r, b_1, \dots, b_s$ in $[n]$ with $r, s \geq \lceil n/3 \rceil$. For each $i \in [\lceil n/3 \rceil]$, we proceed as follows. If $(a_i, b_i) \in A_e$, let $x_i = (a_i, b_i)$ and $y_i = (b_i, b_i)$. Else, $(a_i, b_i) \in \overline{A_e}$, and let $x_i = (a_i, a_i)$ and $y_i = (a_i, b_i)$. Let $X = \{x_1, \dots, x_{\lceil n/3 \rceil}\}$ and $Y = \{y_1, \dots, y_{\lceil n/3 \rceil}\}$. Clearly, $X \subseteq A_e$ and $Y \subseteq \overline{A_e}$. Moreover, X is an independent set, since no two of its vertices share a coordinate, and similarly for Y. Finally, each x_i is adjacent to y_i and no other y_j with $j \neq i$. Therefore, $\text{cutsim}_{R_{n,m}}(A_e, \overline{A_e}) \geq \lceil n/3 \rceil$, contradicting the fact that (T, δ) has sim-width $w < \lceil n/3 \rceil$. □

We conclude with some observations related to $\text{simw}(L(K_n))$. Since $L(K_n)$ contains $L(K_{n/2,n/2})$ as an induced subgraph, Theorem 5.6 implies that $\text{simw}(L(K_n)) \geq \lceil n/6 \rceil$, for $n > 12$. Moreover, since $\text{bw}(K_n) = \lceil \frac{2n}{3} \rceil$ for $n \geq 3$ [140], Theorem 6.7 implies that $\text{simw}(L(G)) \leq \lceil \frac{2n}{3} \rceil$.

Lemma 5.17. *Let $n > 12$ be an integer. Then* $\lceil \frac{n}{6} \rceil \leq \text{simw}(L(K_n)) \leq \lceil \frac{2n}{3} \rceil$.

We expect the value of $\text{simw}(L(K_n))$ to be close to the lower bound $\lceil \frac{n}{6} \rceil$ and leave its determination as an open problem:

Open Problem 3. *Determine the exact value of* $\text{simw}(L(K_n))$.

[1] Notice that the statement of the Lemma contains a typo, as $<$ should be replaced by $\leq$.

Chapter 6

Comparing Width Parameters

This chapter contains joint work with Nick Brettell, Andrea Munaro and Daniël Paulusma: *Comparing Width Parameters on Graph Classes* [28].

6.1 Introduction

In Chapter 3, we defined the notion of equivalence of graph width parameters (see Figure 3.1 for the equivalence of some of the graph width parameters on the class of all graphs). We can refine this notion by restricting to a special graph class $\mathcal{G}$ (hence, not necessarily the class of all graphs). More precisely, two graph width parameters p and q are *equivalent for a graph class* $\mathcal{G}$ if, for every subclass $\mathcal{G}'$ of $\mathcal{G}$, the parameter p is bounded on $\mathcal{G}'$ if and only if q is bounded on $\mathcal{G}'$. This definition leads to a natural research question:

For which graph classes does the relationship between two non-equivalent width parameters p *and* q *change?*

For example, two width parameters p and q might be incomparable in general, but when restricted to some special graph class, one of them could dominate the other. Or p might be more powerful than q in general, but when restricted to some special graph class, p and q could become equivalent.

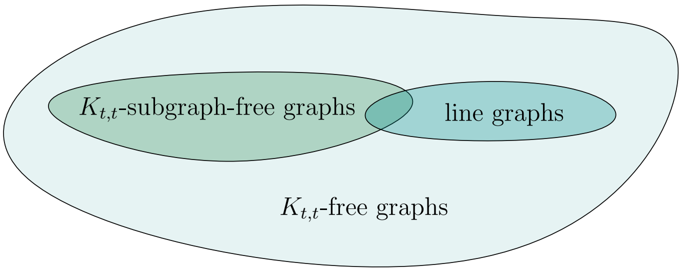

Figure 6.1: Inclusion diagram of the three main graph classes considered in the chapter, where $t \geq 3$.

In this chapter we answer the question above (except for one open case) for the following three graph classes and six width parameters:

- graph classes (see Figure 6.1): $K_{t,t}$-subgraph-free graphs, line graphs, and their common proper superclass (for $t \geq 3$) of $K_{t,t}$-free graphs.
- width parameters (see Figure 3.1): treewidth (tw), clique-width (cw), twin-width (tww), mim-width (mimw), sim-width (simw) and tree-independence number (tree-α).

Apart from the fact that $K_{t,t}$-free graphs (with $t \geq 3$) generalise $K_{t,t}$-subgraph-free graphs and line graphs, our motivation for investigating $K_{t,t}$-free graphs is that large bicliques are obstructions to small tree-independence number. Our main results are summarised in Figures 6.2 to 6.4. They will be explained in detail in Section 6.1.1.

In the following, we recall the results showing that a certain parameter p dominates another parameter q and the corresponding computable functions. It is easy to see that sim-width dominates tree-independence number:

Lemma 6.1. *Let G be a graph. Then* $\text{simw}(G) \leq \text{tree-}\alpha(G)$*.*

Proof sketch. Given a tree decomposition $(F, \{B_t\}_{t \in V(F)})$ of G, the proof of [111, Proposition 3.1] shows how to construct a branch decomposition (T, δ) of G such that, for each $e \in E(T)$, either $N_G(A_e) \cap \overline{A_e}$ or $N_G(\overline{A_e}) \cap A_e$ is contained in a bag in $\{B_t\}_{t \in V(F)}$. Consider then a tree decomposition $(F, \{B_t\}_{t \in V(F)})$ of G with tree-independence number tree-$\alpha(G)$ and the corresponding branch decomposition (T, δ) of G satisfying the property above. Fix $e \in E(T)$ and suppose without loss of generality that $N_G(A_e) \cap \overline{A_e} \subseteq B_t$, for some $t \in V(F)$. This implies that the independence number of $G[N_G(A_e) \cap \overline{A_e}]$ is at most tree-$\alpha(G)$ and so $\text{cutsim}_G(A_e, \overline{A_e}) \leq$ tree-$\alpha(G)$. Since this holds for every $e \in E(T)$, we have that $\text{simw}_G(T, \delta) \leq$ tree-$\alpha(G)$ and so $\text{simw}(G) \leq$ tree-$\alpha(G)$. □

Note also that, by definition, for every graph G,

$$\text{simw}(G) \leq \text{mimw}(G), \tag{6.1}$$

Also, Dallard et al. [48] showed that tree-independence number dominates treewidth: more precisely, for every graph G,

$$\text{tree-}\alpha(G) \leq \text{tw}(G) + 1. \tag{6.2}$$

At last we state a result relating branch width (bw), mm-width (mmw) and treewidth. Vatshelle [150] and Jeong et al. [110] showed that, for every graph G,

$$\text{mmw}(G) \leq \text{bw}(G) \leq \text{tw}(G) + 1 \leq 3\text{mmw}(G), \tag{6.3}$$

so mm-width (mm-width is short for maximum matching width, see [150]) is linearly equivalent to treewidth and branch-width.

6.1.1 Our results

In this section we highlight our main results. We then observe some of their algorithmic and structural consequences and connections with known results.

In Section 6.2 we show the following result concerning $K_{t,t}$-free graphs.

Theorem 6.2. *For every $t \geq 2$, when restricted to $K_{t,t}$-free graphs,*

- *sim-width dominates tree-independence number, tree-independence number and sim-width are more powerful than mim-width, and twin-width is incomparable with these three parameters;*
- *twin-width and mim-width are more powerful than clique-width; and*

- *clique-width is more powerful than treewidth.*

The relationships in Theorem 6.2 are displayed in Figure 6.2. We note from 6.2 that, on $K_{t,t}$-free graphs, tree-independence number becomes more powerful than mim-width, while we do not know yet if tree-independence number dominates sim-width when restricted to $K_{t,t}$-free graphs. If so, then these parameters become equivalent when restricted to $K_{t,t}$-free graphs. This is the *only* missing case in Figure 6.2:

Open Problem 4. *Does tree-independence number dominate sim-width for the class of $K_{t,t}$-free graphs? In other words, is it true that every subclass of $K_{t,t}$-free graphs of bounded sim-width has bounded tree-independence number?*

The main ingredient in the proof of Theorem 6.2 is the following result, which shows in particular that tree-independence number dominates mim-width on $K_{t,t}$-free graphs.

Theorem 6.3. *Let n and m be positive integers. Let G be a $K_{n,m}$-free graph and let (T, δ) be a branch decomposition of G with $\text{mimw}_G(T, \delta) < k$. Then we can construct a tree decomposition of G with independence number less than $6(2^{n+k-1} + mk^{n+1})$ in $O(|V(G)|^{mk^n+4})$ time. In particular,* $\text{tree-}\alpha(G) < 6(2^{n+k-1} + mk^{n+1})$.

This result has structural and algorithmic consequences, as described in Section 6.1.2. In particular, it proves a special case of a conjecture of Dallard, Milanič and Štorgel stating that a hereditary graph class is (tw, ω)-bounded if and only if it has bounded tree-independence number.

In Section 6.3 we consider the class of $K_{t,t}$-subgraph-free graphs, which contains well-known sparse graph classes: for example, every degenerate graph class and every nowhere dense graph class is a (proper) subclass of the class of $K_{t,t}$-subgraph-free graphs for some $t \geq 3$ (see [145] for a proof). Gurski and Wanke [95] proved that for every $t \geq 2$, clique-width and treewidth become equivalent for the class of $K_{t,t}$-subgraph-free graphs. In Section 6.1.2, we use Theorem 6.3 to generalise and extend their result as follows:

Theorem 6.4. *For every $s \geq 3$ and $t \geq 2$, when restricted to $(K_s, K_{t,t})$-free graphs, sim-width, mim-width, clique-width, treewidth and tree-independence number are equivalent, whereas twin-width is more powerful than any of these parameters.*

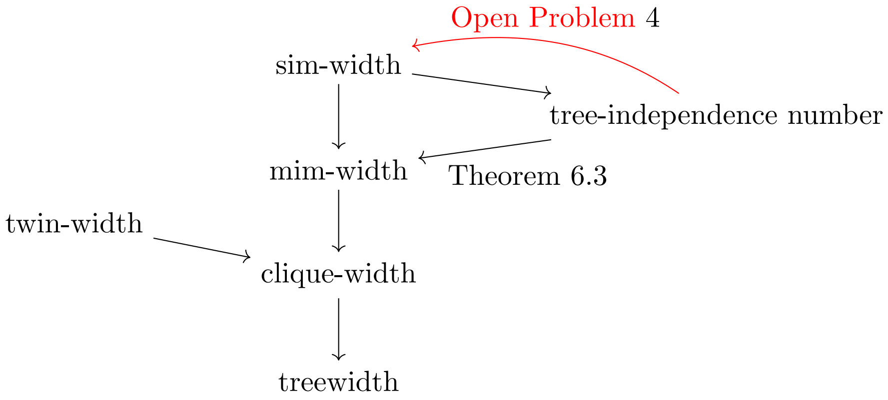

Figure 6.2: The relationships between the different width parameters when restricted to $K_{t,t}$-free graphs for some integer $t \geq 2$. The red arrow illustrates the one remaining open case.

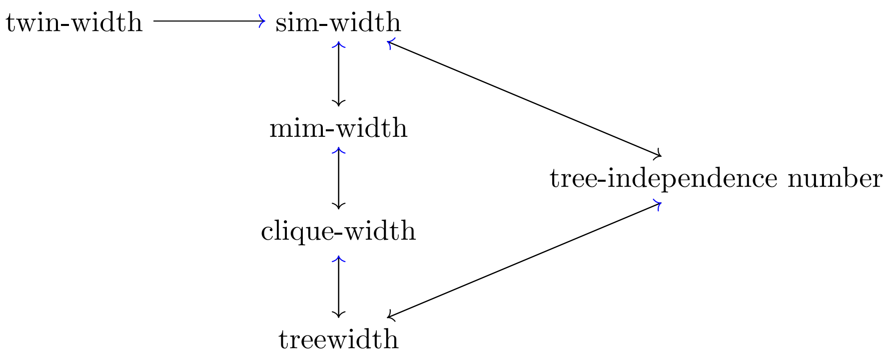

Figure 6.3: The relationships between the different width parameters when restricted to $K_{t,t}$-subgraph-free graphs, for some integer $t \geq 2$, conforming with Theorem 6.3 A bidirectional arrow indicates that the width parameters are equivalent. The same relationships also hold for $(K_s, K_{t,t})$-free graphs; see Theorem 6.4.

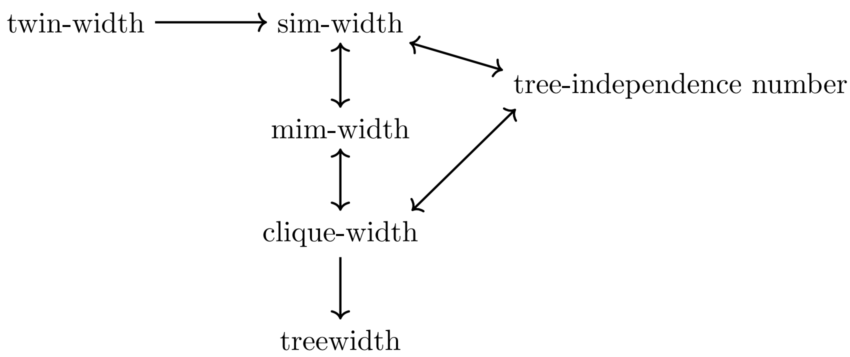

Figure 6.4: The relationships between the six different width parameters when restricted to line graphs. See Theorem 6.6.

The relationships in Theorem 6.4 are displayed in Figure 6.3. Theorem 6.4 shows in particular that treewidth, clique-width, mim-width, sim-width and tree-independence number are equivalent for the class of $(K_s, K_{t,t})$-free graphs. When $s \geq 2t$, the class of $(K_s, K_{t,t})$-free graphs contains the class of $K_{t,t}$-subgraph-free graphs, so these parameters are also equivalent for the class of $K_{t,t}$-subgraph-free graphs. Thus, Theorem 6.4 indeed generalises and extends the result of Gurski and Wanke [95].

Corollary 6.5. *For every $t \geq 2$, when restricted to $K_{t,t}$-subgraph-free graphs, sim-width, mim-width, clique-width, treewidth and tree-independence number are equivalent, whereas twin-width is more powerful than any of these parameters.*

We use Corollary 6.5 in the proof of our next theorem, which concerns line graphs. For a graph class $\mathcal{G}$, we let $L(\mathcal{G})$ denote the class of line graphs of graphs in $\mathcal{G}$. Some years after [95], Gurski and Wanke [96] proved that a class of graphs $\mathcal{G}$ has bounded treewidth if and only if $L(\mathcal{G})$ has bounded clique-width. We extend this result by proving the following theorem in Section 6.4.

Theorem 6.6. *For a graph class $\mathcal{G}$, the following statements are equivalent:*

1. *The class $\mathcal{G}$ has bounded treewidth;*
2. *The class $L(\mathcal{G})$ has bounded clique-width;*
3. *The class $L(\mathcal{G})$ has bounded mim-width;*
4. *The class $L(\mathcal{G})$ has bounded sim-width;*
5. *The class $L(\mathcal{G})$ has bounded tree-independence number.*

Moreover, when restricted to line graphs, sim-width, mim-width, clique-width and tree-independence number are equivalent; twin-width dominates each of these four parameters; and each of the four parameters in turn dominates treewidth.

The main technical contribution towards proving Theorem 6.6 is the following result (Proposition 6.21): There exists a function $f \colon \mathbb{N} \to \mathbb{N}$ such that, for every graph G, $\text{simw}(L(G)) \geq f(\text{tw}(G))$.

Theorem 6.6 shows that, for each parameter $p \in \{\text{cw}, \text{mimw}, \text{simw}, \text{tree-}\alpha\}$, there exist functions f and g such that $f(\text{tw}(G)) \leq p(L(G)) \leq g(\text{tw}(G))$, for every graph G. In fact, the proof shows that we can always choose g as a linear function. However, it is not immediately clear what order of magnitude f should have. For $p = \text{cw}$, Gurski and Wanke [96] showed that a linear function suffices. More precisely, for any graph G, they showed that the following holds:

$$\frac{\text{tw}(G)+1}{4} \leq \text{cw}(L(G)) \leq 2\text{tw}(G)+2. \tag{6.4}$$

But what about the other cases? In Section 6.4, we answer this question for $p = \text{mimw}$. In fact, instead of treewidth, we consider its linearly equivalent parameter branch-width (bw) and show that the mim-width of a line graph equals, up to a multiplicative constant, the branch-width of the root graph:

Theorem 6.7. *For any graph G, $\left\lfloor \frac{\text{bw}(G)}{25} \right\rfloor \leq \text{mimw}(L(G)) \leq \text{bw}(G)$.*

We remark that a result similar in spirit to Theorem 6.7 and Equation (6.4) was obtained by Oum [134]: for any graph G, $\text{rw}(L(G))$ is exactly one of $\text{bw}(G), \text{bw}(G)-1, \text{bw}(G)-2$, where rank-width (rw) is a width parameter equivalent to clique-width.

We conclude this section with a result which will be repeatedly used in our proofs. Recall the definition of wall from Section 4.2.2,

Corollary 6.8. *There exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that the sim-width of the elementary $f(m) \times f(m)$-wall is at least m.*

Proof. It follows from Proposition 5.8 and the fact that the elementary $m \times m$-wall, with $m \geq 7$, has mim-width at least $\sqrt{m}/50$ [26]. □

6.1.2 Consequences of Theorem 6.3

In this section we highlight some structural consequences of Theorem 6.3. Its algorithmic connections with the problem of computing tree decompositions of small independence number will be deferred to Section 6.5, as it is mostly contingent upon some open problems.

Considerable attention has been recently devoted to understanding the substructures of graphs with large treewidth or large pathwidth (a parameter dominated by treewidth). While under the minor and subgraph relations these substructures are well understood thanks to the Grid-minor theorem [139] (see Theorem 6.20) and the Forest-minor theorem [138], much less is known for the induced subgraph relation.

Most results on induced substructures of graphs with large pathwidth or large treewidth deal with specific graph classes such as classes of bounded degree or defined by finitely many forbidden induced subgraphs (see, e.g., [3, 101, 112, 124] and references therein). For other width parameters, the situation is even more obscure. Given a width parameter p, one would like to characterize the families $\mathcal{F}_p$ of unavoidable induced subgraphs of graphs with large p. More formally, $\mathcal{F}_p$ is any set of graphs for which there exists an integer $c \in \mathbb{N}$ such that every graph G with $p(G) > c$ contains a member of $\mathcal{F}_p$ as an induced subgraph[1].

Even though characterizing the families $\mathcal{F}_p$ for fixed $p \in \{\text{cw}, \text{mimw}, \text{simw}, \text{tree-}\alpha\}$ is widely open, there are some obvious graphs that any $\mathcal{F}_p$ must contain. For example, for each of these four parameters p, any $\mathcal{F}_p$ must contain an induced subgraph of every subdivision of a wall and an induced subgraph of the line graph of every subdivision of a wall [26, 47, 48, 96, 111]. Moreover, any $\mathcal{F}_{\text{tree-}\alpha}$ must in addition contain a complete bipartite graph [48]. Observe also that corollary 6.13 can be rephrased as follows. For each $p \in \{\text{cw}, \text{mimw}, \text{simw}, \text{tree-}\alpha\}$ and finitely defined[2] class $\mathcal{C}$ of line graphs, the unavoidable induced subgraphs of graphs in $\mathcal{C}$ with large p are precisely the graphs in $\mathcal{T}$.

Theorem 6.3 readily implies that a graph with large tree-independence number either contains a large complete bipartite graph as an induced subgraph or has large mim-width, and so any $\mathcal{F}_{\text{tree-}\alpha}$ contains precisely some complete bipartite graph and graphs from some $\mathcal{F}_{\text{mimw}}$.

Corollary 6.9. *For every integer $k \geq 1$ and graph G with* $\text{tree-}\alpha(G) \geq 6(2^{2k-1} + k^{k+2})$*, either*

- *G contains a $K_{k,k}$ as an induced subgraph, or*
- $\text{mimw}(G) \geq k$.

Theorem 6.3 has another structural consequence, related to a conjecture of Dallard, Milanič and Štorgel [51]. A graph class $\mathcal{G}$ is (tw, ω)-*bounded* if there exists a function f such that, for every $G \in \mathcal{G}$ and induced subgraph H of G, $\text{tw}(H) \leq f(\omega(H))$. Ramsey's theorem implies that in

[1]The families $\mathcal{F}_{\text{tw}}$ are called *useful families* in [3].
[2]A hereditary class is *finitely defined* if the set of its minimal forbidden induced subgraphs is finite.

every graph class of bounded tree-independence number, the treewidth is bounded by an explicit polynomial function of the clique number, and hence bounded tree-independence number implies (tw, ω)-boundedness [48]. In fact, a partial converse is conjectured to hold [51, Conjecture 8.5]:

Conjecture 6.10 (Dallard, Milanič and Štorgel [51])**.** *A hereditary graph class is* (tw, ω)-*bounded if and only if it has bounded tree-independence number.*

Dallard, Milanič and Štorgel [51] showed that the conjecture holds for every graph class obtained by excluding a single graph H with respect to any of the following containment relations: subgraph, topological minor, minor, and their induced variants. Very recently, Abrishami et al. [1] showed that it holds for (even hole, diamond, pyramid)-free graphs. We use Theorem 6.3 and the fact that a (tw, ω)-bounded graph class is $K_{t,t}$-free for some t [48] to prove that Conjecture 6.10 holds for any (not necessarily hereditary) graph class of bounded mim-width.

Corollary 6.11. *A graph class of bounded mim-width is* (tw, ω)-*bounded if and only if it has bounded tree-independence number.*

Note that there exist (tw, ω)-bounded graph classes of unbounded mim-width, for example chordal graphs or even the proper subclass of split graphs [111].

6.1.3 Consequences of Theorem 6.4

In this section we highlight some algorithmic consequences of Theorem 6.4. With the aid of Theorem 6.4, we can extend the list of NP-hard problems which are in XP parameterized by the sim-width of a given branch decomposition of the input graph. Consider List k-Edge Colouring, where an instance consists of a graph G and a list of colours $L(e) \subseteq \{1, \dots, k\}$ for each $e \in E(G)$, and the task is to determine whether there is an assignment of colours to the edges of G using colours from the lists in such a way that no two adjacent edges receive the same colour. This problem admits a quadratic-time algorithm on every class of bounded tree-independence number, with no tree decomposition required as input [48]. Theorem 6.4 implies the following:

Corollary 6.12. *For every* $k \geq 1$, List k-Edge Colouring *is quadratic-time solvable on every class of bounded sim-width.*

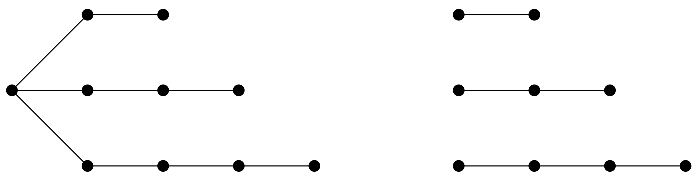

Figure 6.5: The graph $S_{2,3,4} + P_2 + P_3 + P_4$, which is an example of a graph that belongs to $\mathcal{S}$.

Indeed, we simply check whether the input graph of sim-width at most c contains a vertex of degree at least $k+1$. If it does, we have a no-instance. Otherwise, the input graph is $K_{k+1,k+1}$-subgraph-free and, by Corollary 6.5, has tree-independence number at most $f(c,k)$, for some function f (we remark that all bounding functions from Corollary 6.5 are computable). We then apply the algorithm from [48].

6.1.4 Consequences of Theorem 6.6

In this sebsection we explain the consequences of Theorem 6.6. First we require a few definitions. The *claw* is the 4-vertex star with vertices u, v_1, v_2, v_3 and edges uv_1, uv_2 and uv_3. A *subdivided* claw is a graph obtained from a claw by subdividing each of its edges zero or more times. The graph class $\mathcal{S}$ consists of all non-empty disjoint unions of a set of zero or more subdivided claws and paths (see Figure 6.5 for an example of a graph that belongs to $\mathcal{S}$). We let $\mathcal{T}$ consists of all line graphs of graphs in $\mathcal{S}$.

Dabrowski, Johnson and Paulusma [47] showed that for any finite set of graphs $\mathcal{H} = \{H_1, \ldots, H_k\}$, the class of $\mathcal{H}$-free line graphs has bounded clique-width if and only if $H_i \in \mathcal{T}$ for some $i \in \{1, \ldots, k\}$. By using Theorem 6.6 we can extend this result as follows.

Corollary 6.13. *For any finite set of graphs $\mathcal{H} = \{H_1, \ldots, H_k\}$, the following are equivalent:*

- *The class of $\mathcal{H}$-free line graphs has bounded clique-width.*
- *The class of $\mathcal{H}$-free line graphs has bounded mim-width.*
- *The class of $\mathcal{H}$-free line graphs has bounded sim-width.*
- *The class of $\mathcal{H}$-free line graphs has bounded tree-independence number.*
- *$H_i \in \mathcal{T}$ for some $i \in \{1, \ldots, k\}$.*

6.2 $K_{t,t}$-free graphs

In this section, we consider the class of $K_{t,t}$-free graphs and ask which of the following parameters become equivalent or comparable when restricted to this class: sim-width, mim-width, tree-independence number, clique-width, treewidth and twin-width. Recall that their relationships on the class of all graphs are depicted in Figure 3.1. In fact, we answer this question except for one remaining open case (see Open Problem 4), as shown in Figure 6.2.

Let us now analyse the incomparable pairs and check whether they become comparable or equivalent:

Lemma 6.14. *Even when restricted to $K_{2,2}$-free graphs:*

- *Treewidth does not become equivalent to any of the following parameters: sim-width, mim-width, tree-independence number, clique-width and twin-width*
- *Clique-width does not become equivalent to twin-width, mim-width or sim-width;*
- *Mim-width does not become equivalent to sim-width.*

Proof. The class of cliques is $K_{t,t}$-free for any t, has unbounded treewidth but bounded clique-width, twin-width and tree-independence number. Therefore, treewidth does not become equivalent to any of the other parameters. The class of split permutation graphs is $K_{2,2}$-free, has unbounded clique-width [114] but bounded twin-width [21], mim-width [10] and hence sim-width. Therefore, clique-width does not become equivalent to twin-width, mim-width or sim-width. The class of chordal graphs is $K_{2,2}$-free, has unbounded mim-width but sim-width at most 1 [111]. Therefore, mim-width does not become equivalent to sim-width. □

Lemma 6.15. *Even when restricted to $K_{2,2}$-free graphs, twin-width is incomparable with*

- *mim-width,*
- *sim-width, and*
- *tree-independence number.*

Proof. The class of walls is $K_{2,2}$-free, is not (tw, ω)-bounded [48], and hence has unbounded tree-independence number [48], unbounded mim-width [26], and unbounded sim-width by Corollary 6.8, but has bounded twin-width [21]. On the other hand, the class of chordal graphs is $K_{2,2}$-free, has unbounded twin-width [22], but bounded tree-independence number [48], and hence has bounded sim-width. Finally, the class of interval graphs is $K_{2,2}$-free, has unbounded twin-width [22], but bounded mim-width [10]. □

In the rest of this section we show that tree-independence number dominates mim-width on $K_{t,t}$-free graphs (Theorem 6.3). The following two lemmas will be used.

Lemma 6.16. *Let j and ℓ be positive integers. Let G be a graph and let U and V be disjoint subsets of $V(G)$ such that each $u \in U$ has at least one neighbour in V while each $v \in V$ has at most j neighbours in U. If $|U| \geq 2j\ell$, then there exist $X \subseteq U$ and $Y \subseteq V$ such that $|X| = |Y| = \ell$ and $G[X, Y] \cong \ell P_2$.*

Proof. We proceed by induction on ℓ. The base case $\ell = 1$ is trivial. Let $\ell' > 1$ and suppose that the statement holds for each $\ell < \ell'$. We show it holds for ℓ'. Therefore, let $|U| \geq 2j\ell'$. Pick $x \in U$ such that $|N_V(x)|$ is minimum and let $y \in N_V(x)$. Then, at most $j-1$ vertices of $U \setminus \{x\}$ have the same neighbourhood in V as x, or else $|N_U(y)| \geq j+1$, a contradiction. Let U'' be the vertices in $U \setminus \{x\}$ whose neighbourhood in V is distinct from $N_V(x)$. For every vertex $u' \in U''$, we have that $N_V(u') \setminus N_V(x) \neq \varnothing$ by minimality of $|N_V(x)|$. Note that $|U''| \geq |U| - j \geq 2j\ell' - j$. Let $U' = U'' \setminus N_U(y)$. As $|N_U(y)| \leq j$, we have that $|U'| \geq 2j\ell' - j - j = 2j(\ell' - 1)$. Let $V' = V \setminus N_V(x)$. Consider now the graph $G[U' \cup V']$. Each vertex in $U' \subseteq U''$ has at least one neighbour in V' and each vertex in V' has at most j neighbours in U'. Therefore, by the induction hypothesis, there exist $X' \subseteq U'$ and $Y' \subseteq V'$ such that $G[X', Y'] \cong (\ell' - 1)P_2$. Note further that y is anticomplete to $U' = U'' \setminus N_U(y)$ and x is anticomplete to $V' = V \setminus N_V(x)$, so taking $X = X' \cup \{x\}$ and $Y = Y' \cup \{y\}$ completes the proof. □

Lemma 6.17. *Let m be a fixed positive integer. For each positive integer n and k, let $f(n,k) = 2^{n+k}$ and $g_m(n,k) = mk^n$. Let G be a graph and let U and V be disjoint subsets of $V(G)$, where U is an independent set. Suppose, for some positive integers n and k, that $|U| \geq f(n,k)$ and that $\alpha(G[N_V(u)]) \geq g_m(n,k)$ for each $u \in U$. Then, one of the following occurs:*

1. *there exist $X \subseteq U$ and $Y \subseteq V$ such that $|X| = |Y| = k$ and $G[X, Y] \cong kP_2$; or*

2. there exist independent sets $X \subseteq U$ *and* $Y \subseteq V$ *such that* $|X| = n, |Y| = m$ *and* $G[X \cup Y] \cong K_{n,m}$.

Proof. We proceed by double induction on n and k. The base case $n = 1$ or $k = 1$ is an easy exercise left to the reader. Suppose that the statement is true for $(n, k) = (n', k'-1)$, with $n' \geq 1$ and $k' > 1$, and for $(n, k) = (n' - 1, k')$, with $n' > 1$ and $k' \geq 1$. We show that the statement is true for $(n, k) = (n', k')$. Therefore, let $|U| \geq f(n', k')$ and assume $\alpha(G[N_V(u)]) \geq g_m(n', k')$ for each $u \in U$.

Pick $x \in U$ and let $V' = N_V(x)$. Suppose first that there exist at least $f(n' - 1, k')$ vertices $u' \in U \setminus \{x\}$ such that $\alpha(G[N_{V'}(u')]) = \alpha(G[N_V(u') \cap V']) \geq g_m(n' - 1, k')$. Let U' be the set of such vertices. Hence, $|U'| \geq f(n' - 1, k')$. By the induction hypothesis for $(n, k) = (n' - 1, k')$, either there exist $X' \subseteq U'$ and $Y' \subseteq V'$ such that $|X'| = |Y'| = k'$ and $G[X', Y'] \cong k'P_2$, or there exist independent sets $X' \subseteq U'$ and $Y' \subseteq V'$ such that $|X'| = n' - 1$, $|Y'| = m$ and $G[X' \cup Y'] \cong K_{n'-1,m}$. If the former occurs, set $X = X'$ and $Y = Y'$. If the latter occurs, set $X = X' \cup \{x\}$ and $Y = Y'$. In either case, it is easy to see that the statement holds for $(n, k) = (n', k')$ with the chosen X and Y.

Suppose instead that fewer than $f(n' - 1, k')$ vertices $u' \in U \setminus \{x\}$ satisfy $\alpha(G[N_V(u') \cap V']) \geq g_m(n'-1, k')$. This implies that the number of vertices $a \in U \setminus \{x\}$ such that $\alpha(G[N_V(a) \cap V']) < g_m(n' - 1, k')$ is at least $(f(n', k') - 1) - (f(n' - 1, k') - 1) = f(n', k' - 1)$. Let A be the set of such vertices. Hence, $|A| \geq f(n', k' - 1)$. Let now $B = V \setminus V' = V \setminus N_V(x)$. Observe that, for each $a \in A$, we have

$$\begin{aligned}
\alpha(G[N_B(a)]) &= \alpha(G[N_V(a) \setminus V']) \\
&\geq \alpha(G[N_V(a)]) - \alpha(G[N_V(a) \cap V']) \\
&> g_m(n', k') - g_m(n' - 1, k') \\
&> g_m(n', k' - 1).
\end{aligned}$$

By the induction hypothesis for $(n, k) = (n', k' - 1)$, either there exist $X' \subseteq A$ and $Y' \subseteq B$ such that $|X'| = |Y'| = k' - 1$ and $G[X', Y'] \cong (k' - 1)P_2$, or there exist independent sets $X' \subseteq A$ and $Y' \subseteq B$ such that $|X'| = n'$, $|Y'| = m$ and $G[X' \cup Y'] \cong K_{n',m}$. If the latter occurs, set $X = X'$ and $Y = Y'$ and the statement holds for $(n, k) = (n', k')$ with the chosen X and Y. If however the former occurs, first set $X = X' \cup \{x\}$. Observe now that, since x is anticomplete to

$Y' \subseteq B = V \setminus V'$, it is sufficient to find a neighbour of x in V which is anticomplete to X'. Since $x \in U$, we have that $\alpha(G[V']) \geq g_m(n', k')$. Let I be an independent set of $G[V']$ of size at least $g_m(n', k')$. By definition of A, each $x' \in X' \subseteq A$ is such that $\alpha(G[N_V(x') \cap V']) < g_m(n'-1, k')$ and so, for each $x' \in X'$, we have $|N_V(x') \cap I| < g_m(n'-1, k')$. Therefore, since $|X'| = k' - 1$, we have that $|N_V(X') \cap I| < k' g_m(n'-1, k')$. However, $|I| \geq g_m(n', k') = k' g_m(n'-1, k')$, and so $I \setminus N_V(X') \neq \varnothing$. This implies that at least one vertex in V' is anticomplete to X'. Pick such a vertex y and set $Y = Y' \cup \{y\}$. The statement then holds for $(n, k) = (n', k')$ with the chosen X and Y. □

We are finally ready to prove Theorem 6.3, which we restate for convenience.

Theorem 6.3. *Let n and m be positive integers. Let G be a $K_{n,m}$-free graph and let (T, δ) be a branch decomposition of G with* $\text{mimw}_G(T, \delta) < k$*. Then we can construct a tree decomposition of G with independence number less than* $6(2^{n+k-1} + mk^{n+1})$ *in* $O(|V(G)|^{mk^n+4})$ *time. In particular,* $\text{tree-}\alpha(G) < 6(2^{n+k-1} + mk^{n+1})$.

Proof. Let $f(n, k) = 2^{n+k}$ and $g_m(n, k) = mk^n$. We begin by describing an algorithm that constructs a tree decomposition $\mathcal{T}$ of G from the branch decomposition (T, δ) of G. The bags of $\mathcal{T}$ will be constructed recursively. We first need to introduce some notation.

For each $t \in V(T)$ and $i \in \mathbb{N} \cup \{0\}$, we let $X_t|_i$ denote a particular subset of $V(G)$, where i represents the step of the recursion (think of $X_t|_i$ as a bag assigned to t at step i). Given a pair $(T, \{X_t|_i\}_{t \in V(T)})$ satisfying (T1) and (T3) (but not necessarily (T2)) and $v \in V(G)$, we denote by $T_v|_i$ the subtree of T induced by the set $\{t \in V(T) : v \in X_t|_i\}$. Let $e = ab$ be an edge of T. Deleting e splits T into two subtrees, $T(a, b)$ and $T(b, a)$, where $a \in V(T(a, b))$ and $b \in V(T(b, a))$. We say that $T(a, b)$ (resp. $T(b, a)$) *hosts* $v \in V(G)$ if $\delta(v)$ is a leaf of $T(a, b)$ (resp. $T(b, a)$). For $u \in V(G)$, the edge $ab \in E(T)$ *touches* $T_u|_i$ *at* a if $a \in V(T_u|_i)$ and $b \notin V(T_u|_i)$. If the edge $ab \in E(T)$ touches $T_u|_i$ at a, we let

$$N(u, b, a)|_i = \{v \in N_G(u) \setminus X_a|_i : T(b, a) \text{ hosts } v\}.$$

Algorithm. We now describe the algorithm. We first pre-process T by recursively contracting edges having at least one endpoint of degree 2. So we may assume that all internal nodes of T have degree 3. We let i represent a step counter.

Set $i = 0$. Set $X_t|_0 = \{\delta^{-1}(t)\}$ if t is a leaf of T, and set $X_t|_0 = \varnothing$ if t is an internal node of T. For each triple (u, b, a), where $u \in V(G)$ and ab touches $T_u|_0$ at a, compute $N(u, b, a)|_0$.

While there exists a triple (u, b, a), where $u \in V(G)$, the edge ab touches $T_u|_i$ at a, and $N(u, b, a)|_i$ contains an independent set of G of size $g_m(n, k)$, do:

> Pick an arbitrary such triple (u, b, a). Set $X_b|_{i+1} = X_b|_i \cup \{u\}$, label u a *bad vertex with respect to* X_b and, for each $c \neq b$, set $X_c|_{i+1} = X_c|_i$. Compute $N(u, b, a)|_{i+1}$ for each triple (u, b, a) where $u \in V(G)$ and ab touches $T_u|_{i+1}$ at a. Set $i = i + 1$.

While there exists a triple (u, b, a), where $u \in V(G)$, the edge ab touches $T_u|_i$ at a, and $N(u, b, a)|_i \neq \varnothing$, do:

> Pick an arbitrary such triple (u, b, a). Set $X_b|_{i+1} = X_b|_i \cup \{u\}$, label u a *good vertex with respect to* X_b and, for each $c \neq b$, set $X_c|_{i+1} = X_c|_i$. Compute $N(u, b, a)|_{i+1}$ for each triple (u, b, a) where $u \in V(G)$ and ab touches $T_u|_{i+1}$ at a. Set $i = i + 1$.

Return $\mathcal{T} = (T, \{X_t|_i\}_{t \in V(T)})$.

Running time analysis. We now analyse the running time of the algorithm. Let $z = |V(G)|$ so, by definition, z is the number of leaves of T. Since each vertex of T has degree either 1 or 3, the number of internal vertices of T is $z - 2$ and T has at most $2z - 3$ edges. Therefore, at each iteration of the while-do loops, there are $O(z^2)$ triples to be checked. For each such triple (u, b, a), the set $N(u, b, a)|_i$ has size at most z and so checking whether it contains an independent set of G of size $g_m(n, k)$ can be done in $O(z^{g_m(n,k)})$ time. Since computing all the sets $N(u, b, a)|_{i+1}$ takes $O(z^3)$ time, each iteration of the while-do loops can be done in $O(z^{g_m(n,k)+2} + z^3) = O(z^{g_m(n,k)+2})$ time. Observe now that, after each iteration of either loop, if (u, b, a) is the chosen triple, then $T_u|_i$ is extended into $T_u|_{i+1}$ by the addition of one node of T. For each $u \in V(G)$, at most $O(z)$ such extensions are possible and so the total number of iterations of the while-do loops is $O(z^2)$. Therefore, the running time of the algorithm is $O(z^{g_m(n,k)+4})$.

Correctness. We first show that the algorithm indeed outputs a tree decomposition of G. Suppose that we stop at step s, so $N(u,b,a)|_s = \varnothing$ for all triples (u,b,a) such that $u \in V(G)$ and ab touches $T_u|_s$ at a. We show that $\mathcal{T} = (T, \{X_t|_s\}_{t \in V(T)})$ is indeed a tree decomposition of G; namely, it satisfies (T1), (T2) and (T3). Since δ is a bijection from $V(G)$ to the leaves of T and since $\delta^{-1}(t) \in X_t|_s$ for each leaf t of T, (T1) holds. Consider now (T3). Let u be an arbitrary vertex of G. We claim that $T_u|_s$ is connected. Observe that, at each step $i+1$ (for $i \geq 0$), we add u to a bag $X_b|_i$ only if there exists an edge $ab \in E(T)$ that touches $T_u|_i$ at a. This implies that $T_u|_{i+1}$ is obtained from $T_u|_i$ by adding the node b which is adjacent to $a \in T_u|_i$. Hence, if $T_u|_i$ is connected, then $T_u|_{i+1}$ is connected. Since $T_u|_0$ is connected, the same holds for $T_u|_s$. Finally, consider (T2). Suppose (T2) does not hold, so there exists $uv \in E(G)$ such that no bag in $\{X_t|_s\}_{t \in V(T)}$ contains both u and v. Then $T_u|_s$ and $T_v|_s$ share no common nodes. Since T is connected and, by (T3), $T_u|_s$ and $T_v|_s$ are connected subgraphs of T, there is a path in T from $T_u|_s$ to $T_v|_s$. The first edge of this path when traversing from $T_u|_s$ to $T_v|_s$, say ab, touches $T_u|_s$ at a. Then $v \in N(u,b,a)|_s$, so $N(u,b,a)|_s \neq \varnothing$, contradicting that the algorithm terminates at step s.

Finally, we show that $\alpha(\mathcal{T}) < 3f(n,k) + 6g_m(n+1,k)$. Suppose, to the contrary, that $\alpha(\mathcal{T}) \geq 3f(n,k) + 6g_m(n+1,k)$. Then, there exists a bag $X_t \subseteq V(G)$, with $t \in V(T)$, such that $\alpha(G[X_t]) \geq 3f(n,k)+6g_m(n+1,k)$. Observe that t is an internal node of T, as $|X_t| = 1$ for each leaf t of T. Let $P \subseteq X_t$ be an independent set of G such that $|P| \geq 3f(n,k) + 6g_m(n+1,k)$. Since T is a subcubic tree, $T - t$ is the disjoint union of at most three trees. One of these trees, say T_1, must host at least $|P|/3 = f(n,k) + 2g_m(n+1,k)$ vertices of P. Let $P' \subseteq P$ be the vertices of P hosted by T_1. Hence, $|P'| \geq f(n,k) + 2g_m(n+1,k)$. Let $t' \in V(T_1)$ be the unique vertex such that $t't \in E(T)$. Since each vertex of $P' \subseteq X_t$ has been labelled either good or bad with respect to X_t, we have that either at least $f(n,k)$ vertices of P' are bad, or at least $2g_m(n+1,k)$ vertices of P' are good.

Suppose first that at least $f(n,k)$ vertices of P' are bad. Let $U \subseteq P' \subseteq P$ be the set of such vertices and let $V \subseteq V(G)$ be the set of vertices not hosted by T_1. Pick an arbitrary

$u \in U$ and suppose that u has been added to X_t at step $i+1$ for some $i \geq 0$. Then, $\alpha(G[N(u,t,t')|_i]) \geq g_m(n,k)$, where $N(u,t,t')|_i \subseteq N_V(u)$. Therefore, we found an independent set U of G disjoint from V and such that $|U| \geq f(n,k)$ and, for each $u \in U$, we have $\alpha(G[N_V(u)]) \geq \alpha(G[N(u,t,t')|_i]) \geq g_m(n,k)$. Since G is $K_{n,m}$-free, Lemma 6.17 implies that there exist $X \subseteq U$ and $Y \subseteq V$ such that $|X| = |Y| = k$ and $G[X,Y] \cong kP_2$. But then, letting $e = tt'$, we obtain that $\mathrm{cutmim}_G(A_e, \overline{A_e}) \geq k$, contradicting that $\mathrm{mimw}_G(T,\delta) < k$.

Suppose instead that at least $2g_m(n+1,k)$ vertices of P' are good. Let $U = \{u_1, u_2, \dots\} \subseteq P' \subseteq P$ be the set of such vertices. Hence, $|U| \geq 2g_m(n+1,k)$. For each $u_j \in U$, let $i_j + 1$ be the step u_j has been added to X_t, for some $i_j \geq 0$. Let u_1 be the first vertex of U added to X_t, at step $i_1 + 1$. For u_1 to be added to X_t at step i_1+1 as a good vertex, it must be that $N(u_1,t,t')|_{i_1} \neq \varnothing$ but $\alpha(G[N(u_1,t,t')|_{i_1}]) < g_m(n,k)$, for some edge $tt' \in E(T)$ touching $T_{u_1}|_{i_1}$ at t'. Moreover, for every other triple (y,v,v'), where $y \in V(G)$ and vv' touches $T_y|_{i_1}$ at v', it must be that $\alpha(G[N(y,v,v')|_{i_1}]) < g_m(n,k)$.

Now let $V = \bigcup_{u_j \in U} N(u_j,t,t')|_{i_j}$. We claim that each $v \in V$ is adjacent to fewer than $g_m(n,k)$ vertices of U. Suppose, to the contrary, that there exists $v \in N(u_j,t,t')|_{i_j}$, for some $u_j \in U$, which is complete to a subset $U' \subseteq U$ with $|U'| \geq g_m(n,k)$. Then, as $v \in N(u_j,t,t')|_{i_j}$ for some $u_j \in U$, it follows that $v \notin X_{t'}|_{i_j}$. Since $i_j \geq i_1$, this implies that $v \notin X_{t'}|_{i_1}$ and so $T_v|_{i_1}$ does not contain the node t'. Since $v \in N(u_j,t,t')|_{i_j}$, we have that v is hosted by $T(t,t')$. Now let ab be the first edge of the shortest path in T from $T_v|_{i_1}$ to t'. Clearly, ab touches $T_v|_{i_1}$, say without loss of generality at a. Note that $T(t',t)$ is a subtree of $T(b,a)$. Since u_1 is the first vertex of U added to X_t, no other $u_j \in U$ is added to X_t at step i_1. Moreover, since each vertex $u_j \in U$ is hosted by $T(t',t)$, we have that $u_j \notin X_a|_{i_1}$. Therefore, for each $u' \in U'$, it must be that $u' \in N_G(v) \setminus X_a|_{i_1}$. This implies that $U' \subseteq N(v,b,a)|_{i_1}$ and so $\alpha(G[N(v,b,a)|_{i_1}]) \geq \alpha(G[U']) = |U'| \geq g_m(n,k)$. But then, at step i_1+1, the vertex v should have been added to $X_b|_{i_1}$ instead of adding u to $X_t|_{i_1}$, a contradiction.

By the previous paragraph, each $v \in V$ has fewer than $g_m(n,k)$ neighbours in U. Moreover, for each $u_j \in U$, the vertex u_j has been added to X_t and so u_j has a neighbour in V. Therefore, U and V satisfy the conditions of Lemma 6.16 with $j = g_m(n,k)$ and $\ell = k$, and so there exist $X \subseteq U$ and $Y \subseteq V$ such that $|X| = |Y| = k$ and $G[X,Y] \cong kP_2$. Since $U \subseteq T(t',t)$

and $V \subseteq T(t,t')$, by letting $e = tt'$ we have that $\text{cutmim}_G(A_e, \overline{A_e}) \geq k$, contradicting that $\text{mimw}_G(T,\delta) < k$.

To summarize, $\alpha(\mathcal{T}) < 3f(n,k) + 6g_m(n+1,k) = 6(2^{n+k-1} + mk^{n+1})$, as desired. □

Remark 6.18. Bergougnoux et al. [14] introduced o-mim-width, which is a parameter defined in terms of branch decompositions: the definition coincides with that of mim-width except that one side of the maximum matching is required to be an independent set. It is easy to see that o-mim-width is squeezed between mim-width and sim-width. Moreover, it is not difficult to see that the proof above actually shows that tree-independence-number dominates o-mim-width on $K_{n,n}$-free graphs, since one side of the induced matching we obtain in the proof is always an independent set. If we can modify the proof so that both sides of the matching contain large independent set, then this will answer Open Problem 4.

6.3 $K_{t,t}$-subgraph free graphs

In this section we consider $K_{t,t}$-subgraph free graphs. Results from Section 6.2 allow us to prove Theorem 6.4

Theorem 6.4. *For every $s \geq 3$ and $t \geq 2$, when restricted to $(K_s, K_{t,t})$-free graphs, sim-width, mim-width, clique-width, treewidth and tree-independence number are equivalent, whereas twin-width is more powerful than any of these parameters.*

Proof. Let G be a $(K_s, K_{t,t})$-free graph with $\text{simw}(G) = w$. Since G contains no induced subgraph isomorphic to $K_s \boxminus K_s$ and $K_s \boxminus S_s$, Porpoistion 5.8 implies that $\text{mimw}(G) \leq R(R(w+1,s), R(s,s))$. Let $k = R(R(w+1,s), R(s,s))$. Since G is $K_{t,t}$-free, Theorem 6.3 implies that tree-$\alpha(G) \leq 6(2^{t+k-1} + tk^{t+1})$. Since G is K_s-free, [48, Lemma 3.2] implies that $\text{tw}(G) \leq R(s, 6(2^{t+k-1} + tk^{t+1}) + 1) - 2$. Moreover, recall that clique-width always dominates treewidth and that sim-width always dominates clique-width (see Chapter 6). Observe finally that twin-width is not dominated by any of these parameters, even for $(K_3, K_{2,2})$-free graphs. Indeed, walls are $(K_3, K_{2,2})$-free and have bounded twin-width, but each of the other parameters is unbounded, by Corollary 6.8. □

6.4 Line graphs

6.4.1 The proof of Theorem 6.6

In this subsection we start working with line graphs. Recall that Gurski and Wanke [96] showed that for a class of line graphs $\{L(G) : G \in \mathcal{G}\}$, clique-width is equivalent to treewidth for the underlying graph class $\mathcal{G}$. In this section, we show that, in fact, clique-width (and hence treewidth for the underlying graph class) is also equivalent to mim-width, sim-width and tree-independence number.

Due to known results, it suffices to prove that there is a function f such that $\operatorname{simw}(L(G)) \geq f(\operatorname{tw}(G))$ for every graph G; we prove this as Proposition 6.21. Towards this, we require a preliminary result. In [111, Lemma 4.5], it is shown that the sim-width of a graph cannot increase when contracting an edge. In Theorem 5.5, we show that the sim-width of the corresponding line graph cannot increase either. In order to prove Proposition 6.21, we also require the following two results, the first of which is an easy observation (see [111, Lemma 4.5]).

Lemma 6.19. *Let G be a graph and $v \in V(G)$. Then* $\operatorname{simw}(G) \geq \operatorname{simw}(G - v)$.

Theorem 6.20 (Grid-minor theorem [139])**.** *There exists a function $g\colon \mathbb{N} \to \mathbb{N}$ such that, for every $k \in \mathbb{N}$, every graph of treewidth at least k contains the $g(k) \times g(k)$-grid as a minor.*

Proposition 6.21. *There exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for every graph G,* $\operatorname{simw}(L(G)) \geq f(\operatorname{tw}(G))$.

Proof. Let G_n denote the $n \times n$-grid. Let G be a graph with $\operatorname{tw}(G) = k$ and let g be the function from Theorem 6.20. Then G contains $G_{g(k)}$ as a minor. This implies that there exists a finite sequence of operations $\{f_i\}_{i=1}^m$, where each f_i is either an edge contraction or an edge or vertex deletion, such that $f_1(f_2(\cdots f_m(G)\cdots)) \cong G_{g(k)}$. Observe now that, for each operation f_i, $\operatorname{simw}(L(G)) \geq \operatorname{simw}(L(f_i(G)))$, by Theorem 5.5 if f_i is an edge contraction, or by Lemma 6.19 if f_i is an edge or vertex deletion. Therefore, $\operatorname{simw}(L(G)) \geq \operatorname{simw}(L(f_1(f_2(\cdots f_m(G)\cdots)))) = \operatorname{simw}(L(G_{g(k)}))$. But since $L(G_{g(k)})$ is K_5-free, Proposition 5.8 implies that there exists an increasing function $h\colon \mathbb{N} \to \mathbb{N}$ such that

$$h(\operatorname{simw}(L(G_{g(k)}))) \geq \operatorname{mimw}(L(G_{g(k)})). \tag{6.5}$$

Recall now the following well-known fact (see, e.g., [99]): Given a minimum width tree decomposition of $L(G)$, replacing each edge with both of its endpoints gives a tree decomposition of G and so $\mathrm{tw}(L(G)) \geq \frac{1}{2}(\mathrm{tw}(G)+1)-1$. Therefore, since $\mathrm{tw}(G_{g(k)}) = g(k)$, we have that $\mathrm{tw}(L(G_{g(k)})) \geq \frac{1}{2}(g(k)+1)-1$. But the line graph of the $g(k) \times g(k)$-grid is $K_{6,6}$-subgraph-free and so, by the proof of Theorem 6.4, there exists a non-decreasing function $h'\colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{mimw}(L(G_{g(k)})) \geq h'(\frac{1}{3}(\mathrm{tw}(L(G_{g(k)}))+1)) \geq h'(\frac{1}{6}(g(k)+1))$, where the second inequality follows from Equation (6.3). Combining this chain with Equation (6.5), we obtain that there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{simw}(L(G_{g(k)})) \geq f(k)$. This concludes the proof. □

We are finally ready to prove Theorem 6.6, which we restate for convenience.

Theorem 6.6. *For a graph class $\mathcal{G}$, the following statements are equivalent:*

1. *The class $\mathcal{G}$ has bounded treewidth;*
2. *The class $L(\mathcal{G})$ has bounded clique-width;*
3. *The class $L(\mathcal{G})$ has bounded mim-width;*
4. *The class $L(\mathcal{G})$ has bounded sim-width;*
5. *The class $L(\mathcal{G})$ has bounded tree-independence number.*

Moreover, when restricted to line graphs, sim-width, mim-width, clique-width and tree-independence number are equivalent; twin-width dominates each of these four parameters; and each of the four parameters in turn dominates treewidth.

Proof. The implications $2 \Rightarrow 3 \Rightarrow 4$ follow from Figure 3.1. The implication $4 \Rightarrow 1$ follows from Proposition 6.21. The implication $1 \Rightarrow 2$ follows from Equation (6.4), while $5 \Rightarrow 4$ follows again from Figure 3.1. The implication $1 \Rightarrow 5$ follows from the proof of [19, Lemma 2.4] (see also [48, Theorem 3.12]). We provide the short argument for completeness. To this end, let $G \in \mathcal{G}$ and let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ be a tree decomposition of width at most k, for some $k \in \mathbb{N}$. We build a tree decomposition $\mathcal{T}'$ of $L(G)$ as follows: Replace each bag X_t with the set B_t of edges of G incident with a vertex in X_t. It is easy to see that, for each $t \in V(T)$, $\alpha(L(G)[B_t]) \leq |X_t| \leq k+1$. Hence, $\alpha(\mathcal{T}') \leq k+1$. Therefore, $1, 2, 3, 4, 5$ are equivalent.

To complete the proof, it suffices to observe that twin-width is not dominated by any of the other parameters and that treewidth does not dominate any of the other parameters for line graphs (see Theorem 6.6). As for the former, let $\mathcal{G}$ be the class of 1-subdivisions of walls. Then $L(\mathcal{G})$ contains the class of net-walls, which has unbounded treewidth (see, e.g., [26]). However, $L(\mathcal{G})$ consists of planar graphs and hence has bounded twin-width [21]. As for the latter, let $\mathcal{G}$ be the class of stars. Then $L(\mathcal{G})$ coincides with the class of complete graphs, which has unbounded treewidth, but for which any other parameter is bounded. □

6.4.2 The proof of Theorem 6.7

In this subsection we prove Theorem 6.7, which we restate here for convinience:

Theorem 6.7. *For any graph G, $\left\lfloor \frac{\mathrm{bw}(G)}{25} \right\rfloor \leq \mathrm{mimw}(L(G)) \leq \mathrm{bw}(G)$.*

Recall the definition of branch decomposition from Chapter 2. For a branch deompostion of graph G with $S = E(G)$ and $X \subseteq E(G)$, let $\mathrm{mid}(X)$ be the set of vertices that are incident with both an edge in X and another edge in $E(G) \setminus X$, and let $\eta_G(X) = |\mathrm{mid}(X)|$. We define the *branch-width of* G, denoted $\mathrm{bw}(G)$, to be the η_G-branch-width on $E(G)$.

The upper bound of Theorem 6.7 follows from the fact that, for any graph G, $\mathrm{mimw}(G) \leq \mathrm{rw}(G)$ and $\mathrm{rw}(L(G)) \leq \mathrm{bw}(G)$ [133]. We provide a short direct proof for completeness.

Lemma 6.22. *For any graph G, $\mathrm{mimw}(L(G)) \leq \mathrm{bw}(G)$.*

Proof. Let (T, δ) be a branch decomposition of G of minimum η_G-width $\mathrm{bw}(G)$ with $S = E(G)$. Clearly, (T, δ) is a branch decomposition of $L(G)$ as well (with $S = E(G) = V(L(G))$), so it suffices to show that $\mathrm{cutmim}_{L(G)}(A_e) \leq |\mathrm{mid}(A_e)|$ for any $e \in E(T)$. This follows from the fact that an induced matching of size k in $L(G)[A_e, \overline{A_e}]$ provides k distinct vertices in G that are incident with an edge in A_e and another edge in $\overline{A_e}$. □

We now turn to the lower bound.

Proposition 6.23. *Let n be a positive integer and let G be a graph such that $\mathrm{bw}(G) \geq 25n$. Then $\mathrm{mimw}(L(G)) \geq n$.*

Proof. Let (T, δ) be an arbitrary branch decomposition on $V(L(G))$. Then, (T, δ) is also a branch decomposition on $E(G)$. Since $\text{bw}(G) \geq 25n$, there exists $e \in E(T)$ such that $\eta_G(A_e) = |\text{mid}(A_e)| \geq 25n$. It is then enough to show that $\text{cutmim}_{L(G)}(A_e, \overline{A_e}) \geq n$.

Let $M = \text{mid}(A_e)$. Hence, $|M| \geq 25n$. For every vertex $m \in M \subseteq V(G)$, there exist edges $x \in A_e$ and $y \in \overline{A_e}$ of G such that both x and y are incident with m. We can then define two functions, l and r, as follows. The function l assigns to each $m \in M$ a vertex $l(m) \in V(G)$ such that $l(m)m \in A_e$. The function r assigns to each $m \in M$ a vertex $r(m) \in V(G)$ such that $r(m)m \in \overline{A_e}$. Note that l and r are not necessarily injective and that, for each $m \in M$, the vertices m, $l(m)$ and $r(m)$ are pairwise distinct. Given M, l and r as above, a *perfect triple* (L, D, R) is a triple such that $D \subseteq M$, $L = \{l(d) : d \in D\}$, $R = \{r(d) : d \in D\}$ and L, D and R are pairwise disjoint. The *size* of the perfect triple (L, D, R) is $|D|$.

Observe that if there exists a perfect triple (L, D, R) of size n, then $\text{cutmim}_{L(G)}(A_e, \overline{A_e}) \geq n$. Indeed, suppose that $D = \{d_1, \ldots, d_n\}$. By definition, for each $i \in \{1, \ldots, n\}$, we have that $l(d_i)d_i \in A_e$, $r(d_i)d_i \in \overline{A_e}$ and $l(d_i)d_i$ is adjacent to $r(d_i)d_i$ in $L(G)$. Consider now $X = \{l(d_1)d_1, \ldots, l(d_n)d_n\} \subseteq A_e$ and $Y = \{r(d_1)d_1, \ldots, r(d_n)d_n\} \subseteq \overline{A_e}$. Since (L, D, R) is a perfect triple, L, D and R are pairwise disjoint. Therefore, for $i \neq j$, we have that $\{l(d_i), d_i\} \cap \{r(d_j), d_j\} = \varnothing$ and so $L(G)[X, Y] \cong nP_2$. Consequently, $\text{cutmim}_{L(G)}(A_e, \overline{A_e}) \geq n$.

In view of the paragraph above, it is enough to show that there exists a perfect triple (L, D, R) of size at least n. Suppose, to the contrary, that every perfect triple has size less than n. Let (L, D, R) be a perfect triple of maximum size $|D| = k < n$. Since $M \neq \varnothing$ and for each $m \in M$, m, $l(m)$ and $r(m)$ are pairwise distinct, we have that $k \geq 1$. We now consider the reason why (L, D, R) cannot be extended to a larger perfect triple by adding a vertex $m \in M \setminus D$ to D, and possibly $l(m)$ to L and $r(m)$ to R, if not already in L and R, respectively. There are six possible cases:

1. $m \in L \cup R$;
2. $m \notin L \cup R$, $l(m) \in D$ and $r(m) \notin L \cup D$;
3. $m \notin L \cup R$, $l(m) \in R$ and $r(m) \notin L \cup D$;
4. $m \notin L \cup R$, $l(m) \notin R \cup D$ and $r(m) \in D$;

5. $m \notin L \cup R$, $l(m) \notin R \cup D$ and $r(m) \in L$;

6. $m \notin L \cup R$, $l(m) \in R \cup D$ and $r(m) \in L \cup D$.

In the following series of claims, we show that each of these cases holds for a small number of vertices of M.

Claim 6.24. *Case 1 holds for at most $2k$ vertices of M.*

Proof of Claim 6.24. This follows from the fact that $|L| \leq |D|$ and $|R| \leq |D|$, from which $|L \cup R| \leq 2k$. ◇

Claim 6.25. *Cases 2 and 4 each hold for at most $3k$ vertices of M.*

Proof of Claim 6.25. By symmetry, it suffices to consider Case 2. Suppose, to the contrary, that there exist $3k+1$ vertices $m \in M \setminus (L \cup D \cup R)$ satisfying case 2. Since $|D| = k$, the pigeonhole principle implies that there exists $d \in D$ and four vertices $m_1, \ldots, m_4 \in M \setminus (L \cup D \cup R)$ such that $l(m_i) = d$ and $r(m_i) \notin L \cup D$ for each $i \in \{1, \ldots, 4\}$. Then, we remove $l(d), d, r(d)$ from L, D, R, respectively. We claim that we can add d into L, at least two of $m_1, \ldots, m_4$ into D, and the corresponding $r(m_i)$'s into R to obtain a perfect triple of size larger than k. Since $d \notin L \cup R$ and, for each $i \in \{1, \ldots, 4\}$, we have $m_i \notin L \cup R \cup D$ and $r(m_i) \notin L \cup D$, the only possible obstacle is that after adding m_1 to D, and $r(m_1)$ to R, we have that for each m_j with $j \neq 1$, either $m_j = r(m_1)$ (so we cannot add m_j to D), or $r(m_j) = m_1$ (so we cannot add $r(m_j)$ to R). Observe that there exist distinct indices $p, q \in \{2, 3, 4\}$ such that $m_p \neq r(m_1)$ and $m_q \neq r(m_1)$, or else two vertices of m_2, m_3, m_4 coincide. Without loss of generality, $m_2 \neq r(m_1)$ and $m_3 \neq r(m_1)$. We then assume that $r(m_2) = r(m_3) = m_1$, or else we immediately get a perfect triple of size $k+1$. But in this case we can add m_2 and m_3 into D, m_1 into R, and d into L to obtain a perfect triple of size $k+1$, a contradiction. ◇

Claim 6.26. *Cases 3 and 5 each hold for at most $6k$ vertices of M.*

Proof of Claim 6.26. By symmetry, it suffices to consider Case 3. Suppose, to the contrary, that there exist $6k+1$ vertices $m \in M \setminus (L \cup D \cup R)$ satisfying Case 3. Let S be the set of such vertices. For each $b \in R$, let $w_b = |\{d \in D : r(d) = b\}|$ and $w'_b = |\{s \in S : l(s) = b, r(s) \notin L \cup D\}|$.

Note that $\sum_{b\in R} w_b = |D| = k$ and $\sum_{b\in R} w'_b = |S| \geq 6k+1$. Therefore, there exists $c \in R$ such that $w'_c \geq 6w_c + 1$. Let $p = w_c$ and let $M' = \{s \in S : l(s) = c, r(s) \notin L \cup D\}$. Hence, $|M'| = w'_c \geq 6p+1$ and take an arbitrary subset $M'' \subseteq M'$ of size $6p+1$.

We now claim that there exists $Q \subseteq M''$ of size at least $p+1$ such that $r(q) \notin Q$ for every $q \in Q$. To see this, for each $m \in M''$, let $\deg(m) = |\{m'' \in M'' : r(m'') = m\}|$. Observe that $\sum_{m\in M''} \deg(m) \leq |M''| = 6p+1$. So the number of vertices $m \in M''$ with $\deg(m) \geq 2$ is at most $3p$ and there are at least $3p+1$ vertices $m \in M''$ with $\deg(m) \leq 1$. Let $M^* = \{m \in M'' : \deg(m) \leq 1\}$. Hence, $|M^*| \geq 3p+1$. Since each $m \in M^*$ satisfies $\deg(m) \leq 1$, there is at most one $m'' \in M''$ such that $r(m'') = m$. We now show how to construct $Q \subseteq M^*$ such that, for every $q \in Q$, $r(q) \notin Q$. Iteratively, for each $m \in M^*$, add m into Q and possibly remove the following vertices from M^* (in case they belong to M^*): $m, r(m)$ and at most one vertex $m'' \in M''$ such that $r(m'') = m$. At each step, we add one vertex into Q and remove at most three vertices from M^*. Therefore, we can repeat the step above $p+1$ times in order to obtain Q of size at least $p+1$. By construction, for each $q \in Q$, we have $r(q) \notin Q$, as desired.

Let Q be a set given by the previous paragraph. We move c from R to L, remove from D any $d \in D$ with $r(d) = c$ and add Q into D. Moreover, we remove from R any $r \in R$ such that no $d \in D$ satisfies $r(d) = r$, from L any $l \in L$ such that no $d \in D$ satisfies $l(d) = l$, and finally add $\{r(q) : q \in Q\}$ into R. Observe that $Q \subseteq S$ and S is disjoint from $L \cup D \cup R$. Moreover, each $q \in Q \subseteq M'$ satisfies $l(q) = c$ and $r(q) \notin L \cup D$ and, by the previous paragraph, $r(q) \notin Q$. Therefore, we obtain a perfect triple. In this process we have removed $w_c = p$ vertices from D and added $|Q| \geq p+1$ vertices into D, and so we obtained a perfect triple of larger size, a contradiction. This concludes the proof of Claim 6.26. ◇

Claim 6.27. *Case 6 holds for at most $4k+1$ vertices of M.*

Proof of Claim 6.27. Suppose, to the contrary, that there exist $4k+2$ vertices $m \in M \setminus (L \cup D \cup R)$ satisfying Case 6. Let Q be the set of such vertices. Hence, $|Q| \geq 4k+2$. Consider now the multigraph H with vertex set $V(H) = V(G)$ and such that, for each $q \in Q$, $l(q)r(q)$ is an edge of H. Observe that $l(q) \neq r(q)$ for each $q \in Q$, so H does not have loops. It is well known that every loopless multigraph has a bipartite subgraph with at least half of its edges (see, e.g., [151, Theorem 1.3.19]). Let H' be a bipartite subgraph of H, with bipartition (V_1, V_2), and at least $|E(H)|/2 \geq (4k+2)/2 = 2k+1$ edges. Each edge of H' is of the form $e_q = l(q)r(q)$ for some $q \in Q$, and either $l(q) \in V_1$ and $r(q) \in V_2$, or $l(q) \in V_2$ and $r(q) \in V_1$. Without loss of generality, at least $k+1$ edges e_q of H' satisfy the former, and let E_q be the corresponding set. Let $D' = \{q \in Q : e_q \in E_q\}$, $L' = \{l(d) : d \in D'\}$ and $R' = \{r(d) : d \in D'\}$. By construction, $L' \cap R' = \varnothing$. Moreover, for each $q \in D'$, since $q \notin L \cup D \cup R$ and $l(q) \in R \cup D$ and $r(q) \in L \cup D$, we have that $D' \cap (L' \cup R') = \varnothing$. Therefore, (L', D', R') is a perfect triple of size $|D'| \geq k+1$, a contradiction. ◇

By the previous series of claims, there are at most $2k + 2 \cdot 3k + 2 \cdot 6k + (4k+1) = 24k+1$ vertices of $M \setminus D$ satisfying at least one of the six cases. Since $|M \setminus D| > 24n \geq 24(k+1)$, there is at least one vertex $m \in M \setminus D$ satisfying none of the six cases. We add m into D, and possibly $l(m)$ into L and $r(m)$ into R (if not already present) and obtain a perfect triple of size $k+1$, contradicting the maximality of (L, D, R). Therefore, there exists a perfect triple of size at least n, thus concluding the proof of Proposition 6.23. □

Lemma 6.22 and Proposition 6.23 immediately imply Theorem 6.7:

Theorem 6.7. *For any graph G, $\left\lfloor \frac{\mathrm{bw}(G)}{25} \right\rfloor \leq \mathrm{mimw}(L(G)) \leq \mathrm{bw}(G)$.*

The upper bound on $\mathrm{mimw}(L(G))$ is tight in the sense that, for any integer $n \geq 2$, if we let $G = K_{1,n}$, then $\mathrm{mimw}(L(G)) = \mathrm{bw}(G) = 1$. The problem of determining a tight lower bound on $\mathrm{mimw}(L(G))$ in terms of $\mathrm{bw}(G)$ is left open.

6.5 Concluding remarks and open problems

In Theorems 6.2, 6.4 and 6.6, we investigated the relationships between six width parameters (treewidth, clique-width, twin-width, mim-width, sim-width and tree-independence number) when restricted to $K_{t,t}$-free graphs, $K_{t,t}$-subgraph-free graphs and line graphs in order to examine to what extent relationships between non-equivalent width parameters may change. In this way we also extended and generalised several known results from the literature. Moreover, in the case that two parameters become comparable or equivalent on one of these graph classes, we showed how to obtain computable functions witnessing this.

Arguably, the main unresolved problem is the (only) missing case in Figure 6.2, which corresponds to the following question already stated in Section 6.1.

Open Problem 4. *Does tree-independence number dominate sim-width for the class of $K_{t,t}$-free graphs? In other words, is it true that every subclass of $K_{t,t}$-free graphs of bounded sim-width has bounded tree-independence number?*

We first observe a consequence of a positive answer to Open Problem 4. If tree-independence number dominates sim-width for the class of $K_{t,t}$-free graphs then, in order to prove Conjecture 6.10, it suffices to show that every hereditary (tw, ω)-bounded graph class has bounded sim-width. Note also that complete bipartite graphs have sim-width 1 but are not (tw, ω)-bounded.

We now turn to possible algorithmic consequences of Theorem 6.3, and of its extension contingent to a positive answer to Open Problem 4. Computing optimal decompositions for a certain width is in general an NP-hard problem (see [141] and references therein). However, in some cases, there exist exact or approximation algorithms running in FPT or XP time parameterized by the target width. This is exemplified by treewidth, which admits an exact FPT algorithm [17]. Most of the time, it is in fact sufficient to simply obtain an approximate decomposition: we seek an algorithm that, given a graph of width at most k, outputs a decomposition of width at most $f(k)$, for some computable function f.

It is known that rank-width admits an FPT-approximation algorithm [135], with the current best-known result, in terms of running time, being the following. Fomin and Korhonen [81] showed that, for fixed k, there exists an algorithm that, given an n-vertex graph G, in time $2^{2^{O(k)}}n^2$, either decides that $\text{rw}(G) > k$, or outputs a branch decomposition of G of cutrk_G-width at most $2k$. Recently, it was shown that tree-independence number admits an XP-approximation algorithm: Dallard et al. [50] showed that, for fixed k, there exists an algorithm that, given an n-vertex graph G, in time $2^{O(k^2)}n^{O(k)}$, either decides that tree-$\alpha(G) > k$, or outputs a tree decomposition of G with independence number at most $8k$. For other width parameters, such as mim-width and sim-width, it is a well-known open problem to obtain XP-approximation algorithms.

Open Problem 5 (see, e.g., [105, 111])**.** *Does there exist a computable function f and an algorithm A that, for fixed k and given a graph G, in* XP *time parameterized by k, either decides* *that* mimw$(G) > k$ *(or* simw$(G) > k$*), or outputs a branch decomposition of* G *of mim-width (or sim-width) at most* $f(k)$*?*

Therefore, in contrast to algorithms on classes of bounded treewidth, clique-width or tree-independence number, algorithms for graph problems restricted to classes of bounded mim-width (or sim-width) require a branch decomposition of constant mim-width (or sim-width) as part of the input. Obtaining such branch decompositions in polynomial time has been shown possible for several special graph classes (see, for example, [10, 26, 111]).

One may also consider the problem of finding exact and approximation algorithms for computing optimal decompositions for a certain width parameterized by a parameter other than the target width. For example, Bodlaender and Kloks [18] obtained an XP algorithm for computing pathwidth when parameterized by the treewidth of the input graph, and it is not known whether this can be improved to FPT (see, e.g., [94]). Eiben et al. [77] showed that mim-width admits an exact FPT algorithm parameterized by the treewidth and the maximum degree of the input graph, and an exact FPT algorithm parameterized by the treedepth of the input graph. Groenland et al. [94] obtained a polynomial-time algorithm that approximates pathwidth to within a factor of $O(\mathrm{tw}(G)\sqrt{\log \mathrm{tw}(G)})$. Their key observation is that every graph with large pathwidth either has large treewidth or contains a subdivision of a large complete binary tree. This shows how the study of exact and approximation algorithms for computing optimal decompositions for a certain width is related to the study of obstructions to small width.

A straightforward consequence of Theorem 6.3 is the following. Here the *induced biclique number* of a graph G is the largest $t \in \mathbb{N}$ such that G contains $K_{t,t}$ as an induced subgraph.

Corollary 6.28. *There exists an* XP*-approximation algorithm for tree-independence number parameterized by rank-width and induced biclique number.*

Proof. Let G be the input graph, and let t be the induced biclique number of G. We compute a branch decomposition of G of cutrk_G-width at most $2\mathrm{rw}(G)$ in time $2^{2^{O(\mathrm{rw}(G))}}n^2$ [81]. This is also a branch decomposition of G of mim-width less than $2\mathrm{rw}(G)+1$ (see, e.g., [12, Lemma 2.4]). We then run the algorithm from Theorem 6.3 with this branch decomposition in input. It outputs, in $n^{O(t\mathrm{rw}(G)^t)}$ time, a tree decomposition of G with independence number $O(2^{\mathrm{rw}(G)} + t(2\mathrm{rw}(G)+1)^{t+1})$. □

It is not immediately clear whether Corollary 6.28 gives a conditional improved running time compared to the $2^{O(k^2)}n^{O(k)}$ XP-approximation algorithm parameterized by tree-independence number [50] mentioned above. So we pose the following open problem.

Open Problem 6. *For a graph G with induced biclique number t and rank-width* $\mathrm{rw}(G)$*, find an asymptotically tight upper bound on* tree-$\alpha(G)$ *in terms of t and* $\mathrm{rw}(G)$*.*

In a similar vein, we also observe an immediate consequence of a positive answer to Open Problems 4 and 5. Suppose that Open Problem 5 has a positive answer for sim-width. That is, suppose that there exists an XP-approximation algorithm for sim-width parameterized by the target width. If, in addition, an algorithmic version of Open Problem 4 has a positive answer (i.e., suppose that, given a $K_{t,t}$-free graph G and a branch decomposition of G with sim-width at most k, it is possible to compute a tree decomposition of G with independence number at most $g(t,k)$, for some computable function g, in XP time parameterized by k and t), then we can obtain an XP-approximation algorithm for tree-independence number. Indeed, given an input graph G and an integer k, we simply check whether G is $K_{k+1,k+1}$-free. If not, then tree-$\alpha(G) > k$. Otherwise, G is $K_{k+1,k+1}$-free. We now run the algorithm A from Open Problem 5. Algorithm A either decides that $\mathrm{simw}(G) > k$, and so tree-$\alpha(G) > k$, or outputs a branch decomposition of G of sim-width at most $f(k)$, from which we build a tree decomposition of G with independence number at most $g(k)$ in XP time parameterized by k. This would thus provide a different proof of the main result in [50], although one should expect worse running time and approximation factor.

Other natural problems related to Theorems 6.2, 6.4 and 6.6 consist of optimizing the bounding functions obtained therein, as those provided are most likely not optimal. A first attempt in this direction was made in Theorem 6.7, where we showed that, in fact, $\mathrm{mimw}(L(G))$ equals (up to a multiplicative constant) $\mathrm{bw}(G)$, for any graph G. It would be interesting to refine the bounds for the sim-width of a line graph, and a starting point is to improve Proposition 6.21.

Open Problem 7. *Find an asymptotically optimal function f such that* $\mathrm{simw}(L(G)) \geq f(\mathrm{tw}(G))$ *for any graph G.*

Any attempt to solve Open Problem 7 seems to need to avoid the use of the Grid-minor theorem, as the asymptotically optimal function g in the statement is unknown. Another problem in this direction is whether it is possible to improve the bound in Theorem 6.3.

Open Problem 8. *Is the exponential dependency of tree-independence number in mim-width and induced biclique number from Theorem 6.3 necessary?*

Chapter 7

Fractional Tree-Independence-Number-Fragility

This chapter contains joint work with Esther Galby and Andrea Munaro: *Polynomial-time approximation schemes for induced subgraph problems on fractionally tree-independence-number-fragile graphs* [87].

7.1 Introduction

Many optimization problems involving collections of geometric objects in the d-dimensional space are known to admit a polynomial-time approximation scheme (PTAS). Arguably the earliest example of such behavior is the problem of finding the maximum number of pairwise non-intersecting disks or squares in a collection of unit disks or unit squares, respectively [103]. Such subcollection is usually called an *independent packing*. This result was later extended to collections of arbitrary disks and squares and, more generally, fat objects [35, 79]. The reason for the abundance of approximation schemes for geometric problems is that shifting and layering techniques can be used to reduce the problem to small subproblems that can be solved by dynamic programming. In fact, the same phenomenon occurs for graph problems, as evidenced by the seminal work of Baker [7] on approximation schemes for local problems, such as Independent Set, on planar graphs and its generalisations first to apex-minor-free graphs

[78] and further to graphs embeddable on a surface of bounded genus with a bounded number of crossings per edge [93]. The notion of intersection graph allows to jump from the geometric world to the graph-theoretic one. Recall that given a collection $\mathcal{O}$ of geometric objects in $\mathbb{R}^d$, we can consider its *intersection graph*, the graph whose vertices are the objects in $\mathcal{O}$ and where two distinct vertices $O_i, O_j \in \mathcal{O}$ are adjacent if and only if $O_i \cap O_j \neq \varnothing$. An independent packing in $\mathcal{O}$ is then nothing but an independent set in the corresponding intersection graph. Notice that intersection graphs of unit disks or squares are not proper minor-closed, as they contain arbitrarily large cliques. Our motivating question is the following:

Is there any underlying graph-theoretical reason for the existence of the seemingly unrelated PTASes for Independent Set *mentioned above?*

We provide a positive answer to this question that also allows us to further generalise to a framework of maximization problems. Our approach unifies and extends several known polynomial-time approximation schemes on seemingly unrelated graph classes, such as classes of intersection graphs of fat objects in a fixed dimension or proper minor-closed classes. We remark that the similar question of whether there is a general notion under which PTASes using Baker's technique can be obtained was already asked in [93].

Baker's layering technique relies on a form of decomposition theorem for planar graphs that can be roughly summarized as follows. Given a planar graph G and $k \in \mathbb{N}$, the vertex set of G can be partitioned into k possibly empty sets in such a way that deleting any part induces in G a graph of treewidth at most $O(k)$. Moreover, such a partition together with tree decompositions of width at most $O(k)$ of the respective graphs can be found in polynomial time. A statement of this form is typically referred to as a *Vertex Decomposition Theorem* (VDT) [137]. VDTs are known to exist in planar graphs [7], graphs of bounded-genus and apex-minor-free graphs [78], and H-minor-free graphs [55, 58]. However, their existence is in general something too strong to ask for, as is the case of intersection graphs of unit disks or squares and hence fat objects in general. There are then two natural ways in which one can try to relax the notion of VDT. First, we can consider an approximate partition of the vertex set, where a vertex can belong to some constant number of sets. Second, we can look for a width parameter less restrictive than treewidth.

Dvořák [67] pursued the first direction and introduced the notion of efficient fractional treewidth-fragility. A class $\mathcal{G}$ is efficiently fractionally treewidth-fragile in each of the following cases (see, e.g., [72]): $\mathcal{G}$ is subgraph-closed and has strongly sublinear separators and bounded maximum degree, $\mathcal{G}$ is proper minor-closed, or $\mathcal{G}$ consists of intersection graphs of convex objects with bounded aspect ratio in $\mathbb{R}^d$ (for fixed d) and the graphs in $\mathcal{G}$ have bounded clique number. Dvořák [67] showed that Independent Set admits a PTAS on every efficiently fractionally treewidth-fragile class. This result was later extended [70, 72] to a framework of maximization problems including, for example, Max Weight Distance-d Independent Set, Max Weight Induced Forest and Max Weight Induced Matching. However, the notion of fractional treewidth-fragility falls short of capturing classes such as unit disk graphs, as it implies bounded clique number [67].

One can then try to pursue the second direction mentioned above and further relax the notion of efficient fractional treewidth-fragility by considering width parameters *more powerful* than treewidth and algorithmically useful. A natural candidate is *tree-independence number*, a width parameter defined in terms of tree decompositions which is more powerful than treewidth (see Chapter 3), introduced independently by Dallard et al. [49] and Yolov [153]. Several algorithmic applications of boundedness of tree-independence number have been provided, most notably polynomial-time solvability of (c, ψ)-Max Weight Induced Subgraph (informally, the meta-problem of finding a maximum-weight induced subgraph with clique number[1] at most c satisfying a certain CMSO_2 formula ψ) [122], Max Weight Independent $\mathcal{H}$-Packing [49], and its distance-d version, for d even [122]. These are generalisations of problems such as Max Weight Distance-d Independent Set, Max Weight Induced Forest and Max Weight Induced Matching. Investigating the notion of efficient fractional tree-independence-number-fragility (tree-α-fragility for short) was recently suggested in a talk by Dvořák [71], where it was stated that, using an argument from [75], it is possible to show that intersection graphs of balls and cubes in $\mathbb{R}^d$ are fractionally tree-α-fragile.

A successful notion related to fractional treewidth-fragility is the layered treewidth of a graph [63]. Loosely speaking, a graph G has small layered treewidth if there exist a tree decomposition and a layering of G such that the intersection between any bag and layer is small. In

[1] In fact, their formulation requires the chromatic number to be bounded. However, an easy application of Ramsey's theorem shows that, in every graph class of bounded tree-independence number, having bounded clique number is equivalent to having bounded chromatic number.

particular, the union of any constant number of consecutive layers induces a subgraph of small treewidth and hence Baker's technique applies (provided a suitable layering can be found in polynomial time) [68, 69]. Besides being algorithmically interesting, this notion proved useful especially in the context of coloring-type problems (we refer to [66] for additional references). It should be mentioned that classes of bounded layered treewidth include planar graphs and, more generally, apex-minor-free graphs and graphs embeddable on a surface of bounded genus with a bounded number of crossings per edge, amongst others [62]. It can be shown that bounded layered treewidth implies fractional treewidth-fragility (see Section 7.4). Layered treewidth is also related to local treewidth, a notion first introduced by Eppstein [78], and in fact, on proper minor-closed classes, having bounded layered treewidth coincides with having bounded local treewidth (see, e.g., [62]).

7.1.1 Main results

In this chapter, we investigate the notion of efficient fractional tree-α-fragility, which generalises efficient fractional treewidth-fragility and bounded tree-independence number, and show that it answers our motivating question in the positive and allows to unify and extend several known results. More precisely, we provide an approximation meta-theorem which pushes the limits of tractability well beyond the state of the art by extending existing approximation meta-theorems to the broad family of efficiently fractionally tree-α-fragile classes. In this way, we also give a uniform and natural explanation for a large number of algorithmic results. The family of problems covered by our framework belongs to the general meta-problem introduced by Lund and Yannakakis [126] and called Max Induced Π-Subgraph: Given a graph G, the task is to find a maximum-size *induced* subgraph of G satisfying a certain fixed property Π. Loosely speaking, our framework captures the family of problems whose task is to find a *Max-weight sparse* induced subgraph satisfying some fixed *hereditary* property expressible in counting monadic second-order logic (CMSO_2). Counting monadic second order logic is a counting variant of monadic second-order logic (MSO_2), where one is allowed to have atomic formulae expressing that the cardinality of a set is equal to q modulo p, for some integers $p \geq 2$ and $0 \leq q < p$. As we will not directly work with the formalism of CMSO_2, we refer the reader to [43] for an introduction to this subject.

We can finally define the meta-problem called (c, h, ψ)-Max Weight Induced Subgraph, which will be the focus of this chapter. For $h \in \mathbb{N}$, we say that a CMSO_2 formula ψ *expresses an h-near-monotone property* if, for any graph G and any subset $Y \subseteq V(G)$ with $G[Y] \models \psi$, there exists a system $\{R_v \subseteq Y : v \in Y\}$ of subsets of Y such that $v \in R_v$ for each $v \in Y$, each vertex of Y belongs to R_v for at most h vertices $v \in Y$, and $G[Y \setminus \bigcup_{v \in X} R_v] \models \psi$ for each $X \subseteq Y$. The interesting special case $h = 1$ is that of a formula ψ expressing a monotone property, i.e., a formula ψ such that for any graph G and any subsets $X \subseteq Y \subseteq V(G)$, if $G[Y] \models \psi$, then $G[X] \models \psi$. Let ψ be a fixed CMSO_2 formula expressing an h-near-monotone property and let c be a fixed positive integer.

(c, h, ψ)-Max Weight Induced Subgraph

Input: A graph G equipped with a weight function $w\colon V(G) \to \mathbb{Q}_+$.

Task: Find a set $F \subseteq V(G)$ such that:

1. $G[F] \models \psi$,
2. $\omega(G[F]) \leq c$,
3. F is of maximum weight subject to the conditions above,

or conclude that no such set exists.

The meta-problem (c, h, ψ)-Max Weight Induced Subgraph captures several well-known problems, such as Max Weight Independent Set, Max Weight Induced Matching, Max Weight Induced Forest (see Section 7.6 for several other examples). This framework is in fact closely related to those considered in [84] [122], in the context of exact algorithms, and in [70, 72], in the context of approximation algorithms. We briefly highlight the differences. We already mentioned that Lima et al. [122] showed that, for each fixed c and ψ, (c, ψ)-Max Weight Induced Subgraph (which is nothing but the problem obtained from (c, h, ψ)-Max Weight Induced Subgraph by dropping the h-near-monotonicity constraint) can be solved in polynomial time for graphs of bounded tree-independence number. However, as we shall see in Section 7.6, some sort of monotonicity constraint is inevitable in our case. The main difference between the (c, h, ψ)-Max Weight Induced Subgraph framework and those provided in [70, 72] is that ours does not allow to model problems defined in terms of distances. However, this is again inevitable, as the parity of the distances plays a role: For each $\varepsilon > 0$ and fixed odd $d \geq 3$, it is NP-hard to approximate the distance-d version of Independent Set to within a factor of $n^{1/2-\varepsilon}$ for chordal graphs [80], which coincides with the class of graphs with tree-independence number 1 [49]. To partially overcome this, we also consider the framework of Max Weight Independent $\mathcal{H}$-Packing and its distance-d version called Max Weight Distance-d $\mathcal{H}$-Packing (both formally defined in Section 7.6). Recall from Chapter 3, our main results can be summarized as follows (see also Figure 7.1).

(A) For each fixed $c, h \in \mathbb{N}$ and CMSO_2 formula ψ, (c, h, ψ)-Max Weight Induced Subgraph admits a PTAS on every efficiently fractionally tree-α-fragile class.

(B) Max Weight Independent $\mathcal{H}$-Packing admits a PTAS on every efficiently fractionally tree-α-fragile class.

(C) Every class of intersection graphs of fat objects in $\mathbb{R}^d$, for fixed d, is efficiently fractionally tree-α-fragile.

We also introduce the notion of layered tree-independence number, which is a relaxation of layered treewidth and which provides a strengthening of fractional tree-α-fragility (as we explain in the next section), and prove the following.

(D) For each fixed even $p \in \mathbb{N}$, Max Weight Distance-p Packing admits a PTAS on every class of bounded layered tree-independence number[2] and on every class of intersection graphs of fat objects in $\mathbb{R}^d$, for fixed d.

Remark 7.1. Results A, B, D cannot be improved to guarantee EPTASes, unless $\mathsf{FPT} = \mathsf{W}[1]$. Indeed, Marx [127] showed that Independent Set remains $\mathsf{W}[1]$-complete on intersection graphs of unit disks and unit squares.

The main message of our work is that a doubly-relaxed version of a VDT suffices for algorithmic applications and is general enough to hold for several interesting graph classes. In particular, Result A provides an approximation meta-theorem similar to those obtained in [70, 72] but applicable to the substantially broader family of efficiently fractionally tree-α-fragile classes. It should also be noted that there exist several definitions of fatness in the literature and the

[2] Provided that a tree decomposition and a layering witnessing small layered tree-independence number can be computed efficiently.

one we adopt in this chapter slightly generalises that of Chan [35] and is implicitly used in some of the arguments from [100]. Informally, a collection of objects[3] is fat according to our definition if it satisfies a sort of "low-density property": for each r, there is at most a constant number of pairwise non-intersecting objects of size at least r intersecting any region of size r (see Section 7.2 for the formal definition). In fact, our notion of fatness captures that of *low density*, introduced by Har-Peled and Quanrud in [100]. Therefore, Result A also extends some of the local-search-based PTASes from [100] for Max Induced Π-Subgraph, where Π is a hereditary and mergeable[4] property, on intersection graphs of collections of low-density objects in $\mathbb{R}^d$, for fixed d (or, more generally, collections of objects where every subcollection of pairwise non-intersecting objects has low density).

The natural trade-off in extending the tractable families with respect to approximation is that fewer problems will admit a PTAS. In our case this is exemplified by the minimization problem Feedback Vertex Set, which admits no PTAS on unit ball graphs in $\mathbb{R}^3$, unless $\mathsf{P} = \mathsf{NP}$ [83], but admits an EPTAS on disk graphs [123]. In fact, Lokshtanov et al. [123] established a framework for designing EPTASes for a broad class of unweighted vertex-deletion problems on disk graphs including, among others, Feedback Vertex Set (the complement dual of Max Induced Forest) and d-Bounded Degree Vertex Deletion (the complement dual of Max d-Dependent Set, a problem defined in Section 7.6). Previous sporadic PTASes on this class were known only for Vertex Cover [79, 149], Dominating Set [90], Independent Set [35, 79] and Max Clique [20]. Very recently, Dvořák et al. [76], generalising [123], provided EPTASes for the unweighted minimization meta-problem (Induced) Subgraph Hitting on graph classes with polynomial expansion (which, in the case of subgraph-closed classes, is equivalent to having strongly sublinear separators [73]) and on intersection graphs of convex globally fat[5] objects and pseudo-disks. Results A, B and D complement the results from [76, 123], as (c, h, ψ)-Max Weight Induced Subgraph and Max Weight Independent $\mathcal{H}$-Packing capture the *weighted* dual of several problems addressed therein. For example, every problem which consists in computing a subset $S \subseteq V(G)$ of minimum size such that $G - S$ does not

[3]We remark that the objects need not be convex nor similarly-sized, i.e., the ratio of the largest and smallest diameter of the objects need not be bounded by a fixed constant.

[4]A property Π is mergeable if, for any subsets of vertices X, Y which are at distance at least 2, if X and Y each satisfy Π, then $X \cup Y$ satisfies Π.

[5]Global fatness is the "standard" notion of fatness, typically referred to as fatness. Given $k \geq 1$, an object $O \subseteq \mathbb{R}^d$ is k-globally fat if there exist two d-dimensional balls B_{in} and B_{out} with radius R_{in} and R_{out}, respectively, such that $B_{\text{in}} \subseteq O \subseteq B_{\text{out}}$ and $R_{\text{out}} \leq k \cdot R_{\text{in}}$.

contain any graph from a fixed finite family $\mathcal{F}$ as a (induced) subgraph and $G - S$ has bounded clique number. Observe that any fixed finite family $\mathcal{F}$ containing a complete graph satisfies this requirement, and in this way one can obtain problems such as d-Bounded Degree Vertex Deletion, C_k-Hitting and ℓ-Component Order Connectivity.

7.1.2 Overview of the results and organization of the chapter

Fatness. In Section 7.2, we introduce our notion of fatness, called c-fatness, and compare it with three among the most general (in the sense that they apply to arbitrary objects in arbitrary dimensions) notions of fatness from the literature, which we call global fatness[6], local fatness and thickness (see [147]). We show, in particular, that all these notions are equivalent when restricting to convex objects.

Layered and local tree-independence number. In Section 7.3, we begin our study of fractional tree-α-fragility by introducing a subclass of fractionally tree-α-fragile graphs, namely the class of graphs with bounded layered tree-independence number. We obtain the notion of layered tree-independence number by relaxing the successful notion of layered treewidth and show that, besides graphs of bounded layered treewidth, the following classes have bounded layered tree-independence number:

- Intersection graphs of similarly-sized c-fat objects in $\mathbb{R}^2$ (in particular, unit disk graphs);
- Intersection graphs of unit-width rectangles in $\mathbb{R}^2$;
- (Vertex and edge) intersection graphs of paths with bounded horizontal part on a grid (these classes contain some interesting families of string graphs).

As a consequence, we show that graphs in these classes have $O(\sqrt{n})$ tree-independence number and that this is tight up to constant factors. Moreover, we observe that, for minor-closed classes, having bounded layered tree-independence number is equivalent to having bounded local tree-independence number, which in turn is equivalent to excluding an apex graph as a minor, thus extending a characterization of bounded layered treewidth stated in [62]. We also consider the behavior of layered tree-independence number with respect to graph powers and show that

[6]See Footnote 5.

odd powers of graphs of bounded layered tree-independence number have bounded layered tree-independence number and that this does not extend to even powers. This result is crucial for the proof of Result D.

Fractional tree-α-*fragility.* In Section 7.4, we work with Fractional tree-α-fragility and show that every class of bounded layered tree-independence number is fractionally tree-α-fragile (in fact, efficiently fractionally tree-α-fragile, provided that a tree decomposition and a layering witnessing small layered tree-independence number can be computed efficiently). It is then natural to identify necessary conditions for fractional tree-α-fragility. Graphs of low treewidth must have small-size balanced separators (and the converse holds in subgraph-closed classes [74]), whereas graphs of low tree-independence number must have balanced separators with small independence number. Contrary to fractional treewidth-fragility, where sublinear-size balanced separators are needed [67], one should then expect that in the case of fractional tree-α-fragility it is not the size of a separator that has to be small but rather its independence number. Indeed, we show that every fractionally tree-α-fragile class has balanced separators of sublinear independence number. This result implies, unsurprisingly, that 3-regular expanders and intersection graphs of rectangles in the plane are not fractionally tree-α-fragile. Whether Independent Set admits a PTAS on intersection graphs of rectangles in the plane remains a major open problem (see, e.g., [88]).

Intersection graphs of fat objects. In Section 7.5, we investigate families of intersection graphs of fat objects in bounded dimensional spaces. In particular, we show that such families are efficiently fractionally tree-α-fragile (Result C) and that they are closed under taking odd powers. The latter result is used in the proof of Result D.

PTAS frameworks. In Section 7.6, we provide several examples of problems captured by Max Weight Independent $\mathcal{H}$-Packing, its distance-d version and (c, h, ψ)-Max Weight Induced Subgraph. Moreover, we prove Results A, B, D. Given the generality of the frameworks and the broadness of the graph classes to which they are applicable, the running times obtained are typically not competitive. However, in Section 7.6.4, we focus on a specific problem, namely Max Weight Independent Set, and show how tree-independence number arguments can still lead to competitive PTASes for some classes of intersection graphs. Specifically, we obtain PTASes for Max Weight Independent Set for intersection graphs of families of unit disks, unit-width (or, equivalently, unit-height) rectangles, and paths with bounded horizontal part on a grid, which improve or generalise results from [15, 36, 128] mentioned in the next section. These results were all obtained by applying the shifting technique: consider subgraphs whose geometric realizations are contained in narrow strips and exploit properties of such graphs for dynamic programming. We show that the common structural reason that allows fast dynamic programming algorithms on these graphs is in fact boundedness of tree-independence number.

Subexponential-time algorithms. In Section 7.7, we depart from the main topic of the chapter, namely approximation schemes, and note some interesting consequences of Section 7.3 in relation to extract subexponential-time algorithms that can be summarized as follows. There exists a subexponential-time algorithm for Max Weight Distance-d $\mathcal{H}$-Packing, for each fixed even $d \in \mathbb{N}$, on each class of bounded layered tree-independence number (provided a tree decomposition and a layering witnessing this can be computed efficiently). In particular, we obtain a $2^{O(\sqrt{n}\log n)}$-time algorithm for Max Weight Distance-d $\mathcal{H}$-Packing on intersection graphs of similarly-sized c-fat families of objects in $\mathbb{R}^2$. This is related to the seminal work of de Berg et al. [53], who provided $2^{O(\sqrt{n})}$-time algorithms for the *unweighted* version of many problems on intersection graphs of similarly-sized globally fat objects in $\mathbb{R}^d$. For weighted problems, the situation is much more obscure and de Berg and Kisfaludi-Bak [52] asked to determine the complexity of the weighted versions of problems falling in the framework of [53] when restricted to intersection graphs of similarly-sized fat objects in $\mathbb{R}^2$. Our $2^{O(\sqrt{n}\log n)}$ upper bound partially answers this question, as Max Weight Distance-d $\mathcal{H}$-Packing captures the weighted version of several problems from [53].

In Section 7.8, we conclude this chapter with some open questions whose answers allow to identify the applicability limits of our approximation frameworks.

Remark 7.2. All our PTASes for intersection graphs of geometric objects are not robust, i.e., they require a geometric realization to be part of the input.

7.1.3 Other consequences of our work

All problems mentioned in this section are defined in Section 7.1.5.

Disk graphs. Li et al. [121] provided a PTAS for Max $\mathcal{H}_k$-Free Node Set when restricted to disk graphs of bounded heterogeneity[7] or, in other words, intersection graphs of similarly-sized disks. Moreover, they asked whether the assumption of bounded heterogeneity is necessary and what happens to the weighted version of the problem. Results A and C answer the first question in the negative and show that a PTAS for the weighted version can be obtained in a very general setting.

Unit disk graphs. Unit disk graphs are arguably one of the most well-studied graph classes in computational geometry, as they naturally model several real-world problems. Great attention has been devoted to approximation algorithms for Max Weight Independent Set on this class (see, e.g., [104, 131, 148]). To the best of our knowledge, the fastest known PTAS is a $(1-1/k)$-approximation algorithm with running time $O(kn^{4\lceil\frac{2(k-1)}{\sqrt{3}}\rceil})$ [128]. In Section 7.6 we improve on this running time and provide a $(1-1/k)$-approximation algorithm with running time $O(\lceil 3k\rceil n^{3\lceil\frac{3k-1}{2}\rceil+3})$ (note that $\frac{9}{2} < \frac{8}{\sqrt{3}}$). We also remark that a special type of Decomposition Theorem was recently shown to hold for the class of unit disk graphs. A Contraction Decomposition Theorem (CDT) is a statement of the following form: given a graph G, for any $p \in \mathbb{N}$, one can partition the edge set of G into $E_1, \ldots, E_p$ such that contracting the edges in each E_i in G yields a graph of treewidth at most $f(p)$, for some function $f\colon \mathbb{N} \to \mathbb{N}$. CDTs are useful in designing efficient approximation and parameterized algorithms and are known to hold for classes such as unit disk graphs [8] and graphs of bounded genus [56]. Since these classes are efficiently fractionally tree-α-fragile, our results can be seen as providing a different type of relaxed decomposition theorem for them.

Intersection graphs of unit-height rectangles. As observed by Agarwal et al. [4], this class of graphs arises naturally as a model for the problem of labeling maps with labels of the same font size. Improving on [103], they obtained a $(1-1/k)$-approximation algorithm for Max Weight Independent Set on this class with running time $O(n^{2k-1})$. Chan [36] provided a $(1-1/k)$-approximation algorithm with running time $O(n^k)$. Jana et al. [109] provided a PTAS for Max Bipartite Subgraph on intersection graphs of unit squares (and unit disks) and a 2-approximation algorithm for intersection graphs of unit-height rectangles. Moreover, they asked whether the problem admits in fact a PTAS on the latter class. Result A, together with

[7]The heterogeneity of a disk graph is the ratio of the maximum radius to the minimum radius of disks.

the fact that unit-height rectangle graphs are efficiently fractionally tree-α-fragile, answer this question in the positive.

Intersection graphs of paths on a grid. Asinowski et al. [6] introduced the class of *Vertex intersection graphs of Paths on a Grid* (*VPG graphs* for short). A graph G is a *VPG graph* if there exists a collection $\mathcal{P}$ of paths on a grid $\mathcal{G}$ such that $\mathcal{P}$ is in one-to-one correspondence with $V(G)$ and two vertices are adjacent in G if and only if the corresponding paths intersect. It is not difficult to see that this class coincides with the well-known class of string graphs. If every path in $\mathcal{P}$ has at most k *bends*, i.e., 90 degrees turns at a grid-point, the graph is a *B_k-VPG graph.* Golumbic et al. [92] introduced the class of *Edge intersection graphs of Paths on a Grid* (*EPG graphs* for short) which is defined similarly to VPG, except that two vertices are adjacent if and only if the corresponding paths share a grid-edge. It turns out that every graph is EPG [92] and B_k-EPG graphs have been defined similarly to B_k-VPG graphs. Approximation algorithms for Independent Set on VPG and EPG graphs have been deeply investigated, especially when the number of bends is a small constant (see, e.g., [16, 24, 85, 117]). It is an open problem whether Independent Set admits a PTAS on B_1-VPG graphs [16, 117]. Concerning EPG graphs, Bessy et al. [15] showed that the problem admits no PTAS on B_1-EPG graphs, unless $\mathsf{P} = \mathsf{NP}$, even if each path has its vertical segment or its horizontal segment of length at most 1. On the other hand, they provided a PTAS for Independent Set on B_1-EPG graphs where the length of the horizontal part[8] of each path is at most a constant c with running time $O^*(n^{\frac{3c}{\varepsilon}})$, where $O^*(\cdot)$ hides terms polynomial in c and $1/\varepsilon$. In Section 7.6 we extend this result to a PTAS for Max Weight Independent Set on B_k-EPG and B_k-VPG graphs with bounded horizontal part, for any fixed $k \geq 1$.

7.1.4 Relationships between the main graph classes addressed in the chapter

To facilitate the navigation through the main graph classes related to the paper, we depict in Figure 7.1 a full picture of the relationships between these classes and conclude this section by explaining the incomparabilities therein. The terminology we adopt in the following refers to Figure 7.1.

[8] The *horizontal part* of a path is the interval corresponding to the projection of the path onto the x-axis.

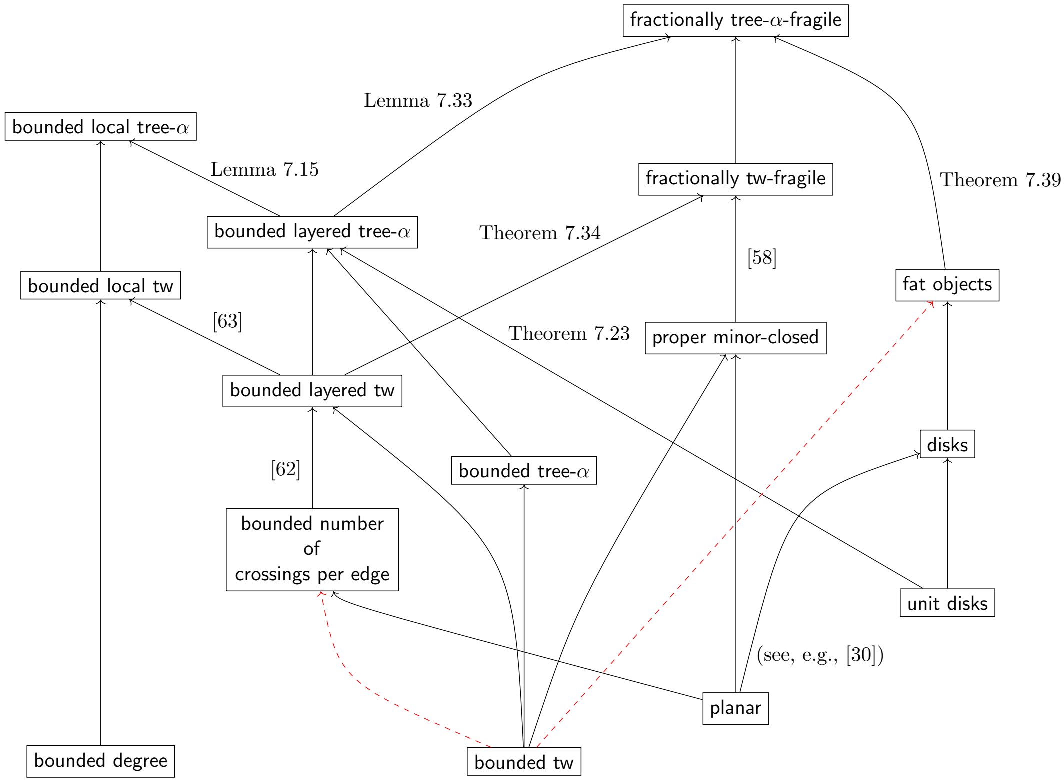

Figure 7.1: Relationships between some of the main graph classes related to this chapter, where an arrow represents class inclusion or implication between class properties, the dashed lines are open cases. The following shorthands are adopted: tw and tree-α are shorthands for treewidth and tree-independence number, respectively. bounded number of crossings per edge stands for the class of graphs embeddable on a surface of bounded genus with a bounded number of crossings per edge. unit disks, disks, and fat objects are shorthands for the class of intersection graphs of a collection of unit disks in the plane, disks in the plane, and fat objects in some d-dimensional space, respectively. The inclusions or implications not directly following from the definition are referenced.

The class of stars shows that bounded tw $\not\subseteq$ bounded degree, planar $\not\subseteq$ unit disks, and planar $\not\subseteq$ bounded degree. The class of complete graphs shows that unit disks $\not\subseteq$ fractionally tw-fragile, unit disks $\not\subseteq$ bounded local tw, bounded tree-α $\not\subseteq$ bounded local tw, and bounded tree-α $\not\subseteq$ fractionally tw-fragile. The class of 2-dimensional grids shows that planar $\not\subseteq$ bounded tree-α and unit disks $\not\subseteq$ bounded tree-α. Theorem 7.17 shows that proper minor-closed $\not\subseteq$ bounded local tree-α. Corollary 7.37 shows that bounded degree $\not\subseteq$ fractionally tree-α-fragile. Corollary 7.44 shows that disks $\not\subseteq$ bounded local tree-α. [93, Theorem 3.4] shows that bounded number of crossings per edge $\not\subseteq$ proper minor-closed. We are not aware of any class showing that bounded tw $\not\subseteq$ bounded number of crossings per edge or that bounded tw $\not\subseteq$ fat objects.

7.1.5 Encoding of geometric intersection graphs

In this short section we explain how the geometric realizations of some of the intersection graphs addressed in this chapter are encoded. A box is *axis-aligned* if its faces are each perpendicular to one of the coordinate axes of the space.

Intersection graphs of unit disks and unit-width rectangles A collection of unit disks is a family of closed disks in $\mathbb{R}^2$ with common radius $c \in \mathbb{R}$, whereas a collection of unit-width rectangles is a family of axis-aligned closed rectangles in $\mathbb{R}^2$ with common width $c \in \mathbb{R}$. A collection of unit disks is encoded by a collection of points in $\mathbb{R}^2$ representing the centers of the disks. Each rectangle in a collection of unit-width rectangles is encoded by the four points in $\mathbb{R}^2$ representing its four vertices. In general, and unless otherwise stated, when we refer to a rectangle we mean an axis-aligned closed rectangle in $\mathbb{R}^2$.

VPG and EPG graphs Given a rectangular grid $\mathcal{G}$, its horizontal lines are referred to as *rows* and its vertical lines as *columns*. For a VPG (EPG) graph G, the pair $\mathcal{R} = (\mathcal{G}, \mathcal{P})$ is a *VPG representation* (*EPG representation*) of G. More generally, a *grid representation* of a graph G is a triple $\mathcal{R} = (\mathcal{G}, \mathcal{P}, x)$ where $x \in \{e, v\}$, such that $(\mathcal{G}, \mathcal{P})$ is an EPG representation of G if $x = e$, and $(\mathcal{G}, \mathcal{P})$ is a VPG representation of G if $x = v$. Note that, irrespective of whether $x = e$ (i.e., G is an EPG graph) or $x = v$ (i.e., G is a VPG graph), if two vertices $u, v \in V(G)$ are adjacent in G then P_u and P_v share at least one grid-point. A *bend-point* of a path $P \in \mathcal{P}$ is a grid-point corresponding to a bend of P and a *segment* of P is either a vertical or horizontal line segment in the polygonal curve constituting P. Paths in $\mathcal{P}$ are encoded as follows. For each $P \in \mathcal{P}$, we have one sequence $s(P)$ of points in $\mathbb{R}^2$: $s(P) = (x_1, y_1), (x_2, y_2), \ldots, (x_{\ell_P}, y_{\ell_P})$ consists of the endpoints (x_1, y_1) and (x_{ℓ_P}, y_{ℓ_P}) of P and all the bend-points of P in their order of appearance when traversing P from (x_1, y_1) to (x_{ℓ_P}, y_{ℓ_P}). If each path in $\mathcal{P}$ has number of bends polynomial in $|V(G)|$, then the size of this data structure is polynomial in $|V(G)|$. Given $s(P)$, we can easily determine the horizontal part $h(P)$ of the path P (i.e., the projection of P onto the x-axis). Note that our results for VPG and EPG graphs (Theorems 7.25 and 7.56 and Corollary 7.29), although stated for constant number of bends, still hold for polynomial (in $|V(G)|$) number of bends, with a worse polynomial running time.

7.2 Comparing different notions of fatness

In this section we introduce our notion of fatness, called c-fatness, and compare it with alternative notions of fatness from the literature. We then show that all these notions are equivalent when restricting to convex objects.

We first require some additional definitions. Let $d \geq 2$ be an arbitrary but fixed integer. Recall that an object in $\mathbb{R}^d$ is a path-connected compact set $O \subset \mathbb{R}^d$. The *size* of an object O in $\mathbb{R}^d$, denoted $s(O)$, is the side length of its smallest enclosing axis-aligned hypercube. Unless otherwise stated, all boxes considered in the chapter are axis-aligned.

Chan [35] introduced the following definition of fatness: A collection of objects in $\mathbb{R}^d$ is fat if, for any r and size-r box R, we can choose a constant number c of points in $\mathbb{R}^d$ such that every object that intersects R and has size at least r contains at least one of the chosen points. In particular, if a collection of objects is fat according to this definition, every size-r box intersects at most c pairwise disjoint objects of size at least r from the collection. Chan also stated that collections of balls and collections of boxes with bounded aspect ratios are fat (recall that the aspect ratio of a box is the ratio of its largest side length over its smallest side length). We slightly generalise this fatness definition as follows.

Definition 7.3. Let $c \in \mathbb{R}$ be a constant. A collection $\mathcal{O}$ of objects in $\mathbb{R}^d$ is *c-fat* if, for any $r \in \mathbb{R}$ and any closed box R of side length r, there exist at most c pairwise non-intersecting objects from $\mathcal{O}$ of size at least r and which intersect R.

Loosely speaking, a collection of objects is fat according to the previous definition if it satisfies a sort of "low-density property". In fact, our notion of fatness captures that of low density, defined by Har-Peled and Quanrud [100] as follows. Given a constant ρ, a collection $\mathcal{O}$ of objects in $\mathbb{R}^d$ has *density* ρ if any object R (not necessarily in $\mathcal{O}$) intersects at most ρ objects from $\mathcal{O}$ with diameter at least that of R. It should be noticed that, in some of the their arguments, Har-Peled and Quanrud make implicit use of the notion of ρ-fatness (see [100, Lemma 3.6]). As mentioned, collections of balls and collections of boxes with bounded aspect ratios in $\mathbb{R}^d$ are c-fat, for some $c \geq 1$; we will formally show this below. As another interesting example, we will also show that certain collections of annuli in $\mathbb{R}^d$ are c-fat.

Remark 7.4. When working with a c-fat collection of objects, we assume that some reasonable operations can be done in constant time: determining center, size and diameter of an object, as well as its projection onto one of the axes, deciding if two objects intersect, and constructing the geometric realization of the collection.

Several definitions of fatness have been proposed in the literature, essentially all of which are equivalent for convex objects. In the rest of this section, we compare the notion of c-fatness defined above with three among the most general (in the sense that they apply to arbitrary objects in arbitrary dimensions) notions of fatness. We refer the reader to [146] for a general overview on fatness.

Let $d \geq 2$ be an arbitrary but fixed integer. For a measurable set $A \subseteq \mathbb{R}^d$, we denote by vol(A) the Lebesgue measure of A. The following definition is due to van der Stappen et al. [147]. Given $k > 0$, an object $O \subseteq \mathbb{R}^d$ is *k-locally fat*[9] if, for each closed d-dimensional ball B with center in O and whose boundary intersects O (or, equivalently, not properly containing O), $\text{vol}(B) \leq k^d \cdot \text{vol}(O \cap B)$. It can be seen that there are no k-locally fat objects for $k < 2$ and balls are exactly the 2-locally fat objects [147]. The terminology is justified by the fact that a constant portion of the proximity of every point of the object must be covered by the object and so no object with infinitesimally thin protuberances is locally fat.

The following two fatness definitions have a more global nature, with the first one being arguably the "standard" definition. Given $k \geq 1$, an object $O \subseteq \mathbb{R}^d$ is *k-globally fat* if there exist two d-dimensional balls B_{in} and B_{out} with radius R_{in} and R_{out}, respectively, such that $B_{\text{in}} \subseteq O \subseteq B_{\text{out}}$ and $R_{\text{out}} \leq k \cdot R_{\text{in}}$. The second definition is again due to van der Stappen et al. [147]. Given $k \geq 1$, an object $O \subseteq \mathbb{R}^d$ is *k-thick* if, denoting by B_O the minimal enclosing ball of O (i.e., the d-dimensional ball B_O of smallest volume such that $O \subseteq B_O$), we have that $\text{vol}(B_O) \leq k^d \cdot \text{vol}(O)$. Clearly, balls are exactly the 1-globally fat objects and are 1-thick.

Observe that a k-locally fat (or k-globally fat, or k-thick) object is k'-locally fat (or k'-globally fat, or k'-thick) for any $k' \geq k$. Moreover, in all these fatness definitions, the value of k can be seen as a qualitative measure of fatness: the smaller the value of k, the fatter the object. The three definitions above naturally extend to collections of objects as follows: Given k, we say that

[9] Notice that, in [147], such a notion is referred to as k-fatness.

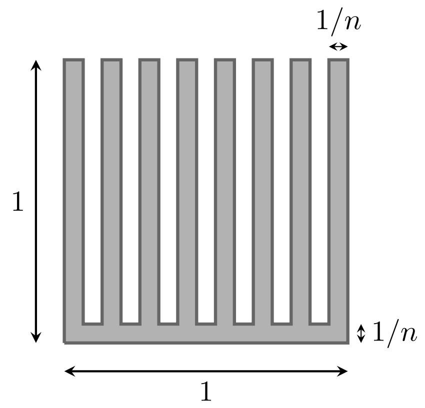

Figure 7.2: A comb O_n with $(n+1)/2$ teeth each of width $1/n$, for some odd $n \in \mathbb{N}$. It is easy to see that O_n is $\sqrt{\pi}$-thick but the collection $\{O_n : n \in \mathbb{N}\}$ is not k-globally fat, for any $k \geq 1$.

a collection of objects in $\mathbb{R}^d$ is *k-locally fat* (or *k-globally fat*, or *k-thick*) if every object in the collection is k-locally fat (or k-globally fat, or k-thick).

It is easy to see that, in general, global fatness is a stronger notion than thickness, as we show next for completeness. Note that, in this section, we make repeated use of the following well-known fact: For any d-dimensional ball B with radius r, $\text{vol}(B) = C_d \cdot r^d$, for a constant C_d depending on d.

Lemma 7.5 (Folklore)**.** *Every k-globally fat object in $\mathbb{R}^d$ is k-thick. Moreover, there exists a k-thick collection of objects in $\mathbb{R}^2$ which is not k'-globally fat, for any $k' \geq 1$.*

Proof. Let O be a k-globally fat object in $\mathbb{R}^d$. By assumption, there exist $k \geq 1$ and two d-dimensional balls B_{in} and B_{out} with radius R_{in} and R_{out}, respectively, such that $B_{\text{in}} \subseteq O \subseteq B_{\text{out}}$ and $R_{\text{out}} \leq k \cdot R_{\text{in}}$. Therefore, $\text{vol}(B_{\text{out}}) \leq k^d \cdot \text{vol}(B_{\text{in}})$. Denoting by B_O the minimal enclosing ball of O, we then obtain that $\text{vol}(B_O) \leq \text{vol}(B_{\text{out}}) \leq k^d \cdot \text{vol}(B_{\text{in}}) \leq k^d \cdot \text{vol}(O)$, and so O is k-thick. As for the second statement, simply consider the collection of combs depicted in Figure 7.2. $\square$

We now observe how our notion of c-fatness relates to local fatness, global fatness and thickness. We first show that c-fatness does not imply thickness and hence global fatness either. An *annulus* is a closed region in $\mathbb{R}^d$ bounded by two concentric d-dimensional balls. The bigger ball is called the *outer ball*, whereas the other is called the *inner ball*.

Lemma 7.6. *Fix $R > 0$. The collection of annuli in $\mathbb{R}^d$ with outer balls of radius R is c-fat, for some $c \geq 1$, but not k-thick, for any $k \geq 1$.*

Proof. The fact that the collection of annuli with outer balls of radius R is c-fat, for some $c \geq 1$, follows from the fact that two such annuli are disjoint if and only if the corresponding outer balls are. Then, from [146, Theorem 2.9] (see also [136, Theorem 2.2]), the number of such objects are bounded. Consider now the remaining statement. Fix $k \geq 1$ and take $n > k$. Consider an annulus A with outer ball of radius R and inner ball of radius $(1 - 1/n^d)^{1/d}R$. The minimal enclosing ball of A has volume $C_d \cdot R^d$, whereas $\mathrm{vol}(A) = C_d \cdot \frac{R^d}{n^d}$. The choice of n implies that A is not k-thick. □

We now show that global fatness (and hence thickness) does not imply c-fatness. The following result comes in handy. Recall that an *outerstring graph* is the intersection graph of a set of curves in $\mathbb{R}^2$ that lie inside a disk such that each curve intersects the boundary of the disk in one of its endpoints.

Lemma 7.7. *Let $k > 1$. Every outerstring graph can be realized as the intersection graph of a k-globally fat collection of objects in $\mathbb{R}^d$, for any $d \geq 2$.*

Proof. We show the result for $d = 2$ and leave the easy extension to the reader. Fix an arbitrary $1 < k < \sqrt{2}$. Let G be an outerstring graph on n vertices. By possibly rescaling, we may assume that its geometric realization consists of the disk B_0 centered at $(1/2, 0)$ and with radius $1/2$. Fix an arbitrary counterclockwise order $a_1, \ldots, a_n$ of the endpoints of the curves on the boundary of B_0. For each positive integer i, let $r_i = (\frac{1}{k-1})^{2i}$ and $c_i = (\frac{1}{k-1})^{2i-1} + (\frac{1}{k-1})^{2i}$, and let B_i be the disk with radius r_i centered at $(c_i, 0)$. Hence, $B_1, \ldots, B_n$ are disks of increasing radius arranged along the positive x-axis. It is easy to see that B_1 is disjoint from B_0. Observe now that, for $i \geq 1$, any point in B_i has x-coordinate at most $c_i + r_i = (\frac{1}{k-1})^{2i-1} + (\frac{1}{k-1})^{2i} + (\frac{1}{k-1})^{2i}$, whereas any point in B_{i+1} has x-coordinate at least $c_{i+1} - r_{i+1} = (\frac{1}{k-1})^{2i+1}$. Since $1 < k < \sqrt{2}$, we have that $(\frac{1}{k-1})^{2i+1} > (\frac{1}{k-1})^{2i-1} + (\frac{1}{k-1})^{2i} + (\frac{1}{k-1})^{2i}$, and so B_{i+1} lies completely to the right of B_i for each $i \geq 0$, hence the disks $B_0, B_1, \ldots, B_n$ are pairwise disjoint. It is easy to see that, for every $i \in \{1, \ldots, n\}$, we can connect a_i to B_i using a thin string S_i in such a way that S_i does not intersect any disk other than B_0 and B_i, S_i is contained in the disk with center $(c_i, 0)$ and radius c_i, and the strings $S_1, \ldots, S_n$ are pairwise disjoint (see Figure 7.3).

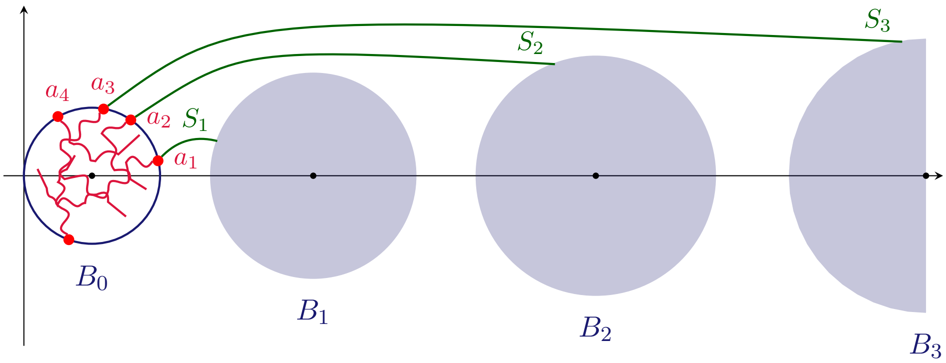

Figure 7.3: Realizing an outerstring graph as the intersection graph of a globally fat collection of objects in $\mathbb{R}^2$.

Let $X_1, \ldots, X_n$ be the geometric objects obtained in this way. Clearly, G is isomorphic to the intersection graph of $\{X_1, \ldots, X_n\}$. By construction, X_i is contained in a disk of radius c_i and contains a disk of radius r_i. Since

$$\frac{c_i}{r_i} = \frac{(\frac{1}{k-1})^{2i-1} + (\frac{1}{k-1})^{2i}}{(\frac{1}{k-1})^{2i}} = \frac{1 + \frac{1}{k-1}}{\frac{1}{k-1}} = k,$$

the collection $\{X_1, \ldots, X_n\}$ is k-globally fat. Since this holds for any $1 < k < \sqrt{2}$, we conclude by recalling that a k-globally fat collection is k'-globally fat for any $k' \geq k$. □

In order to show the following consequence of Lemma 7.7 we make use of two results, Lemma 7.36 and Theorem 7.39, which will be proved in Section 7.4 and Section 7.5, respectively.

Corollary 7.8. *The class of intersection graphs of k-globally fat objects in $\mathbb{R}^d$ is fractionally tree-α-fragile if and only if $k = 1$. In particular, there exists a k-globally fat collection of objects in $\mathbb{R}^d$, for some $k > 1$, which is not c-fat, for any $c \geq 1$.*

Proof. The class of intersection graphs of 1-globally fat objects in $\mathbb{R}^d$, which coincides with the class of intersection graphs of balls in $\mathbb{R}^d$, is fractionally tree-α-fragile thanks to Theorem 7.39. It is easy to see that the class of complete bipartite graphs is contained in that of outerstring graphs. By Lemma 7.7, the latter is contained in the class of intersection graphs of k-globally fat objects in $\mathbb{R}^d$, for any $k > 1$. It then suffices to observe that the class of complete bipartite graphs is not fractionally tree-α-fragile (by Lemma 7.36) and that, for any $c \geq 1$, the class of intersection graphs of c-fat collections of objects in $\mathbb{R}^d$ is fractionally tree-α-fragile (Theorem 7.39). □

Finally, we observe that local fatness is a stronger notion than c-fatness.

Lemma 7.9. *If $\mathcal{O}$ is a k-locally fat collection of objects in $\mathbb{R}^d$, then $\mathcal{O}$ is $(2k)^d$-fat. Moreover, for any fixed $R > 0$, the collection of annuli in $\mathbb{R}^d$ with outer balls of radius R is c-fat, for some $c \geq 1$, but not k-locally fat, for any $k \geq 1$.*

Proof. The first statement immediately follows from [146, Theorem 2.9] (see also [136, Theorem 2.2]). As for the second, it follows from Lemma 7.6 and the observation that every k-locally fat object is k-thick, which is left as an easy exercise. □

We note some consequences of Theorem 7.9. Since balls in any dimension are 2-locally fat, a collection of balls in $\mathbb{R}^d$ is 4^d-fat. Moreover, since the length of a main diagonal of a d-dimensional box of side length l is $l\sqrt{d}$, a size-r box with aspect ratio at most t has volume at least $(\frac{r}{t\sqrt{d}})^d$ and so is $td\sqrt[d]{C_d}$-locally fat (see, e.g., [146, Theorem 2.7]). This implies that a collection of boxes (not necessarily axis-aligned) in $\mathbb{R}^d$ with aspect ratios at most t is $C_d(2td)^d$-fat.

We conclude this section by considering the case of collections of convex objects, where all definitions of fatness introduced above turn out to be equivalent. It is known that, when restricting to convex objects in $\mathbb{R}^d$, local fatness implies global fatness [136, Lemma 4.6], and that local fatness and thickness are equivalent [147, Theorem 2.5]. In order to add c-fatness to the picture, it is enough to show that, for collections of convex objects, c-fatness implies global fatness. In the next two results, we assume that the collections of convex objects are closed under rotations and translations.

Lemma 7.10. *If $\mathcal{O}$ is a c-fat collection of convex objects in $\mathbb{R}^d$, then $\mathcal{O}$ is $cd\sqrt{d}$-globally fat.*

Proof. Let $\{e_1, \dots, e_d\}$ be the standard basis of $\mathbb{R}^d$ and let $O \in \mathcal{O}$ be of size r. We first claim that, for each $i \in \{1, \dots, d\}$, there exist points $p_i, q_i \in O$ such that $\overrightarrow{p_i q_i} = \frac{2r}{c} e_i$. Suppose, to the contrary, that there exists $i \in \{1, \dots, d\}$ such that, for every pair of points $p_i, q_i \in O$, we have that $\overrightarrow{p_i q_i} \neq \frac{2r}{c} e_i$. Since O is convex, it is not difficult to see that this implies that O is contained in a box of width less than $\frac{2r}{c}$ along the i-th axis. But then we can find a subcollection of at least $c+1$ pairwise non-intersecting objects of size r which intersect a fixed box of side length r, as depicted in Figure 7.4, contradicting the assumption that $\mathcal{O}$ is c-fat.

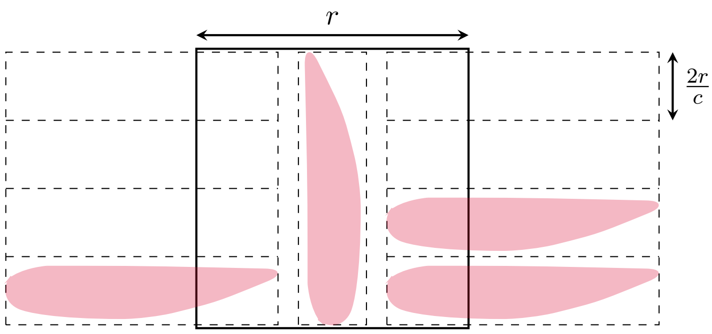

Figure 7.4: A construction used in the proof of Lemma 7.10.

We now show that there exists a point $t \in O$ such that $t + \frac{2r}{cd} e_i \in O$ for every i. This immediately follows from a general claim we show next. For two points p and q in $\mathbb{R}^n$, we denote by the vector from p to q as $\overrightarrow{pq}$.

Claim 7.11. *Let O be a convex object in $\mathbb{R}^d$ and let $\varepsilon > 0$. Suppose that there exists a set of vectors $\{e_1, \dots, e_k\}$ such that, for each $i \in \{1, \dots, k\}$, there exist points $p_i, q_i \in O$ with $\overrightarrow{p_i q_i} = \varepsilon e_i$. Then there exists a point $t \in O$ such that $t + \frac{\varepsilon}{k} e_i \in O$ for every $i \in \{1, \dots, k\}$.*

Proof of Claim 7.11. We proceed by induction on k. The base case $k = 1$ trivially holds. Let now $k > 1$ and suppose that there exists a set of vectors $\{e_1, \dots, e_k\}$ such that, for each $i \in \{1, \dots, k\}$, there exist points $p_i, q_i \in O$ with $\overrightarrow{p_i q_i} = \varepsilon e_i$. By the induction hypothesis, we can find $s \in O$ such that $s + \frac{\varepsilon}{k-1} e_i \in O$ for every $i \in \{1, \dots, k-1\}$. Consider now $t = \frac{k-1}{k} s + \frac{1}{k} p_k$. Since O is convex and $s, p_k \in O$, we have that $t \in O$. Moreover, since $s + \frac{\varepsilon}{k-1} e_i \in O$ for each $i \in \{1, \dots, k-1\}$, we have that $t + \frac{\varepsilon}{k} e_i = \frac{k-1}{k}(s + \frac{\varepsilon}{k-1} e_i) + \frac{1}{k} p_k \in O$ for each $i \in \{1, \dots, k-1\}$. Similarly, since $p_k + \varepsilon e_k = q_k \in O$, it follows that $t + \frac{\varepsilon}{k} e_k = \frac{k-1}{k} s + \frac{1}{k}(p_k + \varepsilon e_k) \in O$. ◇

The above implies that O contains a hypercube of side length $\frac{r}{cd}$ and so a ball of radius $\frac{r}{2cd}$ as well. On the other hand, since O has size r, the side length of its smallest enclosing hypercube is at most r and so the radius of its smallest enclosing ball is at most $r\sqrt{d}/2$. This implies that O is $cd\sqrt{d}$-globally fat. □

Proposition 7.12. *Let $\mathcal{O}$ be a collection of convex objects in $\mathbb{R}^d$. The following are equivalent:*

1. *$\mathcal{O}$ is k_0-fat for some $k_0 \geq 1$;*
2. *$\mathcal{O}$ is k_1-globally fat for some $k_1 \geq 1$;*

3. $\mathcal{O}$ is k_2-thick for some $k_2 \geq 1$;

4. $\mathcal{O}$ is k_3-locally fat for some $k_3 \geq 2$.

Proof. The implication (1) $\Rightarrow$ (2) follows from Lemma 7.10. The implication (2) $\Rightarrow$ (3) follows from Lemma 7.5. The implication (3) $\Rightarrow$ (4) follows from [147, Theorem 2.5]. Finally, the implication (4) $\Rightarrow$ (1) follows from Theorem 7.9. □

7.3 Layered and local tree-independence number

In this section we initiate a study of the notion of layered tree-independence number. Recall the definition of tree-independence number from Chapter 3, a *layering* of a graph G is a partition $(V_0, V_1, \ldots)$ of $V(G)$ such that, for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leq 1$. Each set V_i is a *layer*. The *layered width* of a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of a graph G is the minimum integer ℓ for which there exists a layering $(V_0, V_1, \ldots)$ of G such that, for each bag X_t and layer V_i, we have $|X_t \cap V_i| \leq \ell$. The *layered treewidth* of a graph G is the minimum layered width of a tree decomposition of G. Layerings with one layer show that the layered treewidth of G is at most $\mathrm{tw}(G) + 1$. We now introduce the analogue of layered treewidth for the width parameter tree-independence number.

Definition 7.13. The *layered independence number* of a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of a graph G is the minimum integer ℓ for which there exists a layering $(V_0, V_1, \ldots)$ of G such that, for each bag X_t and layer V_i, we have $\alpha(G[X_t \cap V_i]) \leq \ell$. The *layered tree-independence number* of a graph G is the minimum layered independence number of a tree decomposition of G.

Layerings with one layer show that the layered tree-independence number of G is at most tree-$\alpha(G)$. Moreover, the layered tree-independence number of a graph is clearly at most its layered treewidth. The proof of [63, Lemma 10] shows, mutatis mutandis, that graphs of bounded layered tree-independence number have $O(\sqrt{n})$ tree-independence number, as we observe next.

Lemma 7.14. *Let $k \in \mathbb{N}$ and let G be a n-vertex graph. Given a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of G and a layering $(V_0, V_1, \ldots)$ of G such that, for each bag X_t and layer*

V_i, $\alpha(G[X_t \cap V_i]) \leq k$, it is possible to compute, in time polynomial in n and $|V(T)|$, a tree decomposition of G with independence number at most $2\sqrt{kn}$. In particular, every n-vertex graph with layered tree-independence number k has tree-independence number at most $2\sqrt{kn}$.

Proof. Let $p = \lceil \sqrt{n/k} \rceil$. For each $j \in \{0, \ldots, p-1\}$, let $W_j = V_j \cup V_{p+j} \cup V_{2p+j} \cup \cdots$. Observe that $(W_0, W_1, \ldots, W_{p-1})$ is a partition of $V(G)$. We then find $j \in \{0, \ldots, p-1\}$ such that $|W_j| \leq \frac{n}{p} \leq \sqrt{kn}$. Now, each component K of $G - W_j$ is contained within $p-1$ consecutive layers and so, since $\alpha(G[X_t \cap V_i]) \leq k$ for each bag X_t and layer V_i, restricting the bags in $\mathcal{T}$ to $V(K)$ gives a tree decomposition of K with independence number at most $k(p-1) \leq \sqrt{kn}$. We then merge the tree decompositions of the components of $G - W_j$ into a tree decomposition of $G - W_j$ with independence number at most $\sqrt{kn}$. Finally, adding W_j to every bag of this tree decomposition gives a tree decomposition of G with independence number at most $\sqrt{kn} + |W_j| \leq 2\sqrt{kn}$. □

Given a width parameter p, a graph class $\mathcal{G}$ has *bounded local* p if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every integer $r \in \mathbb{N}$, graph $G \in \mathcal{G}$, and vertex $v \in V(G)$, the subgraph $G[N^r[v]]$ has p-width at most $f(r)$. In [63], it is shown that if every graph in a class $\mathcal{G}$ has layered treewidth at most ℓ, then $\mathcal{G}$ has bounded local treewidth with $f(r) = \ell(2r+1) - 1$. Similarly, bounded layered tree-independence number implies bounded local tree-independence number, as we show next.

Lemma 7.15. *If every graph in a class $\mathcal{G}$ has layered tree-independence number at most ℓ, then $\mathcal{G}$ has bounded local tree-independence number with $f(r) = \ell(2r+1)$.*

Proof. Let $r \in \mathbb{N}$, $G \in \mathcal{G}$ and $v \in V(G)$. Let $G' = G[N^r[v]]$. By assumption, G has a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of layered independence number ℓ with respect to some layering $(V_0, V_1, \ldots)$. Suppose that $v \in V_i$. Then $V(G') \subseteq V_{i-r} \cup \cdots \cup V_{i+r}$ and so, for each bag X_t, we have that $\alpha(G'[X_t]) \leq \sum_{j=-r}^{r} \alpha(G[X_t \cap V_{i-j}]) \leq \ell(2r+1)$. This implies that tree-$\alpha(G') \leq \ell(2r+1)$. □

Corollary 7.16. *The layered tree-independence number of $K_{n,n}$ is at least $n/5$.*

Proof. Suppose, to the contrary, that the layered tree-independence number of $K_{n,n}$ is less than $n/5$. Since the diameter of $K_{n,n}$ is 2, Theorem 7.15 implies that tree-$\alpha(K_{n,n}) < n$, contradicting the fact that tree-$\alpha(K_{n,n}) = n$ [49]. □

In general, bounded local tree-independence number does not imply bounded layered tree-independence number. This will be a consequence of Lemma 7.33 and Corollary 7.37. However, it is known that a proper minor-closed class has bounded layered treewidth if and only if it has bounded local treewidth if and only if it excludes some apex graph[10] as a minor (see, e.g., [63]). The next result extends this equivalence to layered tree-independence number and local tree-independence number.

Theorem 7.17. *The following are equivalent for a minor-closed class $\mathcal{G}$:*

1. *Some apex graph is not in $\mathcal{G}$;*
2. *$\mathcal{G}$ has bounded local tree-independence number;*
3. *$\mathcal{G}$ has linear local tree-independence number (i.e., $f(r)$ is linear in r);*
4. *$\mathcal{G}$ has bounded layered tree-independence number.*

Proof. Consider first (1) $\Rightarrow$ (4). By [63], if $\mathcal{G}$ excludes some apex graph as a minor, then $\mathcal{G}$ has bounded layered treewidth, hence bounded layered tree-independence number as well. The implication (4) $\Rightarrow$ (3) follows from Theorem 7.15. The implication (3) $\Rightarrow$ (2) follows by definition. Finally, consider (2) $\Rightarrow$ (1). Let G_n be the graph obtained from the $n \times n$-grid graph (the Cartesian product of two n-vertex paths) by adding a dominating vertex v_n. Observe that the class $\{G_n : n \in \mathbb{N}\}$ has unbounded local tree-independence number, as v_n is dominating and the class of grids has unbounded tree-independence number (since it is not (tw, ω)-bounded, see [49, Lemma 3.2]). Hence, if $\mathcal{G}$ contains all apex graphs, then in particular it contains $\{G_n : n \in \mathbb{N}\}$ and so has unbounded local tree-independence number. □

Theorem 7.17 implies the following result from [51].

Corollary 7.18. *A minor-closed class has bounded tree-independence number if and only if some planar graph is not in the class.*

[10]An apex graph is a graph that can be made planar by deleting a single vertex.

Proof. Let $\mathcal{G}$ be a minor-closed class. If $\mathcal{G}$ has bounded tree-independence number then, since the class of walls has unbounded tree-independence number [48, 49], $\mathcal{G}$ does not contain a planar graph as a minor.

Conversely, we claim that, for every planar graph H there is an integer c such that every H-minor-free graph G has tree-independence number at most c. Let H^+ be the apex graph obtained from H by adding a dominating vertex v and let G^+ be the graph obtained from G by adding a dominating vertex x. It is shown in [63] that G^+ is H^+-minor-free. By Theorem 7.17, G^+ has layered tree-independence number at most ℓ, for some fixed integer ℓ. Since G^+ has radius 1, at most three layers are used. Thus G^+, and hence G, have tree-independence number at most 3ℓ. □

We now consider the behavior of layered tree-independence number with respect to graph powers. Bonomo-Braberman and Gonzalez [23] showed that fixed powers of bounded treewidth and bounded degree graphs are of bounded treewidth. More specifically, for any graph G and $p \geq 2$, $\text{tw}(G^p) \leq (\text{tw}(G) + 1)(\Delta(G) + 1)^{\lceil \frac{p}{2} \rceil} - 1$. It follows from the work of Dujmović et al. [65] that powers of graphs of bounded layered treewidth and bounded maximum degree have bounded layered treewidth. The upper bound was improved by Dujmović et al. [66], who showed that if G has layered treewidth k, then G^p has layered treewidth less than $2pk\Delta(G)^{\lfloor \frac{p}{2} \rfloor}$. Lima et al. [122] showed that, for any graph G and odd $p \in \mathbb{N}$, tree-$\alpha(G^p) \leq$ tree-$\alpha(G)$ and that, for every fixed even $p \in \mathbb{N}$, there is no function f such that tree-$\alpha(G^p) \leq f($tree-$\alpha(G))$ for all graphs G. We show that odd powers of bounded layered tree-independence number graphs have bounded layered tree-independence number and that this result does not extend to even powers. Before doing so, we need a definition and a result from [122]. Given a graph G and a family $\mathcal{H} = \{H_j\}_{j \in J}$ of subgraphs of G, we denote by $G(\mathcal{H})$ the graph with vertex set J, in which two distinct elements $i, j \in J$ are adjacent if and only if H_i and H_j either have a vertex in common or there is an edge in G connecting them.

Lemma 7.19 (Lima et al. [122])**.** *Let G be a graph and let k and d be positive integers. For $v \in V(G)$, let H_v be the subgraph of G induced by the vertices at distance at most d from v, and let $\mathcal{H} = \{H_v\}_{v \in V(G)}$. Then G^{k+2d} is isomorphic to $G^k(\mathcal{H})$.*

Theorem 7.20. *Let G be a graph and let d be a positive integer. Given a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of G and a layering $(V_1, \ldots, V_m)$ of G such that, for each bag X_t and layer*

V_i, $\alpha(G[X_t \cap V_i]) \leq k$, it is possible to compute in $O(|V(T)| \cdot (|V(G)| + |E(G)|))$ time a tree decomposition $\mathcal{T}' = (T, \{X'_t\}_{t \in V(T)})$ of G^{1+2d} and a layering $(V'_1, \ldots, V'_{\lceil \frac{m}{1+2d} \rceil})$ of G^{1+2d} such that, for each bag X'_t and layer V'_i, $\alpha(G^{1+2d}[X'_t \cap V'_i]) \leq (1+4d)k$. In particular, if G has layered tree-independence number k, then G^{1+2d} has layered tree-independence number at most $(1+4d)k$.

Proof. Let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ and $(V_1, \ldots, V_m)$ be the given tree decomposition and layering of G, respectively. For each vertex $u \in V(G)$, let $l(u)$ be the unique index i such that $u \in V_i$. For each $v \in V(G)$, let H_v be the subgraph of G induced by the vertices at distance at most d from v, and let $\mathcal{H} = \{H_v\}_{v \in V(G)}$. Let $\mathcal{T}' = (T, \{X'_t\}_{t \in V(T)})$, with $X'_t = \{v \in V(G) : V(H_v) \cap X_t \neq \varnothing\}$ for each $t \in V(T)$. By [49, Lemma 6.1], $\mathcal{T}'$ is a tree decomposition of $G(\mathcal{H})$ and hence, by Lemma 7.19, of G^{1+2d} as well. Moreover, for each $v \in V(G)$, $V(H_v) \cap X_t \neq \varnothing$ if and only if v is at distance at most d from X_t in G and the set of all such vertices v can be computed using BFS in $O(|V(G)|+|E(G)|)$ time. Therefore, $\mathcal{T}'$ can be computed in $O(|V(T)| \cdot (|V(G)|+|E(G)|)$ time. For each $1 \leq i \leq \lceil \frac{m}{1+2d} \rceil$, let now $V'_i = \bigcup_{(1+2d)(i-1) < j \leq (1+2d)i} V_j$. We claim that $(V'_1, \ldots, V'_{\lceil \frac{m}{1+2d} \rceil})$ is a layering of G^{1+2d}. Clearly, these sets partition $V(G^{1+2d})$. Moreover, for each edge $uv \in E(G^{1+2d})$, we have that $d_G(u,v) \leq 1+2d$ and so $|l(i) - l(j)| \leq 1+2d$. Consequently, u and v belong to either the same V'_i or to consecutive V'_i's.

We now show that, for each $1 \leq i \leq \lceil \frac{m}{1+2d} \rceil$ and $t \in V(T)$, $\alpha(G^{1+2d}[V'_i \cap X'_t]) \leq (1+4d)k$. Suppose, to the contrary, that $\alpha(G^{1+2d}[V'_i \cap X'_t]) > (1+4d)k$ for some i and t as above. Then there exists an independent set $U = \{u_1, \ldots, u_{(1+4d)k+1}\}$ of G^{1+2d} contained in $V'_i \cap X'_t$. By construction, $V(H_{u_p}) \cap X_t \neq \varnothing$ for each $1 \leq p \leq (1+4d)k+1$ and, for each such p, we pick an arbitrary vertex in $V(H_{u_p}) \cap X_t$ and denote it by $r(u_p, t)$. Note that, for $p \neq q$, $r(u_p, t)$ is distinct from and non-adjacent to $r(u_q, t)$ in G, for otherwise either H_{u_p} and H_{u_q} share a vertex or there is an edge connecting them in G, from which $u_p u_q \in G(\mathcal{H}) = G^{1+2d}$, contradicting the fact that U is an independent set of G^{1+2d}. Now, by construction, each $r(u_p, t)$ belongs to $V(H_{u_p})$ and so is at distance at most d in G from u_p. Moreover, u_p belongs to the layer V'_i of G^{1+2d}, for each $1 \leq p \leq (1+4d)k+1$. Therefore, $(1+2d)(i-1) < l(u_p) \leq (1+2d)i$ and $(1+2d)(i-1) - d < l(r(u_p,t)) \leq (1+2d)i + d$, for each $1 \leq p \leq (1+4d)k+1$. That is, each $r(u_p, t)$ belongs to one of the $1+4d$ consecutive layers $V_{(1+2d)(i-1)-d+1}, \ldots, V_{(1+2d)i+d}$ of

G. Since $|U| > (1 + 4d)k$, at least one such layer V_r contains $k + 1$ vertices of the form $r(u_p, t)$. Therefore, $\alpha(G[V_r \cap X_t]) > k$, a contradiction. □

Lemma 7.21. *Fix an even $k \in \mathbb{N}$. There exist graphs G with tree-independence number 1 and such that the layered tree-independence number of G^k is arbitrarily large.*

Proof. By the proof of [122, Proposition 3.7], for every graph H, there exists a chordal graph G such that G^k contains an induced subgraph isomorphic to H. Take $H = K_{5n,5n}$ and one such G. By Corollary 7.16, the layered tree-independence number of G^k is at least n, whereas tree-$\alpha(G) = 1$. □

7.3.1 Intersection graphs with bounded layered tree-independence number

In this section we show that the following graph classes have bounded layered tree-independence number: intersection graphs of similarly-sized c-fat families of objects in $\mathbb{R}^2$ (in particular, unit disk graphs), unit-width rectangle graphs, and VPG/EPG graphs where the paths have bounded horizontal part and number of bends. As it will appear from the proofs, our tree decompositions witnessing this are in fact path decompositions. Lemma 7.14 then implies that graphs from these classes have $O(\sqrt{n})$ tree-independence number and we argue that this is tight up to constant factors.

Note that, in general, intersection graphs of disks or rectangles in the plane and of paths on a grid (VPG/EPG graphs) all have unbounded layered tree-independence number (see Figure 7.5). This follows from the fact that large bicliques and large grids with a dominating vertex have large layered tree-independence number, thanks to Corollary 7.16 and the proof of Theorem 7.17, respectively.

A collection of objects is *k-similarly-sized* if the ratio of the largest and smallest object diameter is at most some absolute constant $k \geq 1$.

Theorem 7.22. *Let G be the intersection graph of a k-similarly-sized c-fat family $\mathcal{O}$ of n objects in $\mathbb{R}^2$, for some constants c and k. It is possible to compute, in $O(n \log n)$ time, a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ and a layering $(V_1, V_2, \ldots)$ of G such that $|V(T)| = O(n)$*

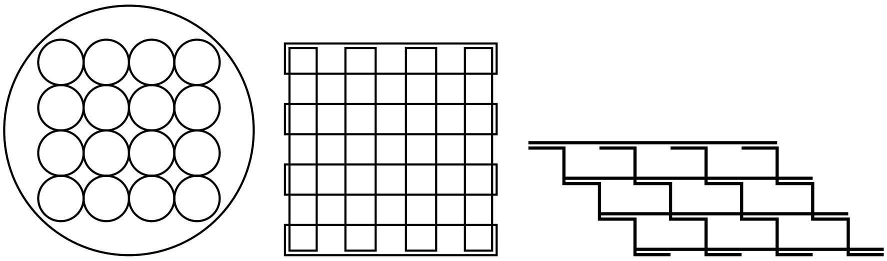

Figure 7.5: Examples showing that intersection graphs of disks and rectangles in $\mathbb{R}^2$ and VPG/EPG graphs have unbounded layered tree-independence number: Realization of the 4×4-grid graph with a dominating vertex as a disk graph (left), and of $K_{4,4}$ as an intersection graph of rectangles (middle) and as a VPG/EPG graph (right).

and, for each bag X_t and layer V_i, $\alpha(G[X_t \cap V_i]) \leq \lceil 2\sqrt{2}k \rceil c$. In particular, G has layered tree-independence number at most $\lceil 2\sqrt{2}k \rceil c$.

Proof. Let $d_{\min}$ and $d_{\max}$ be the minimum and maximum diameter of the objects in $\mathcal{O}$, respectively. Since $\mathcal{O}$ is k-similarly-sized, $d_{\max} \leq k \cdot d_{\min}$. Without loss of generality, the family $\mathcal{O}$ is contained in the positive quadrant. For each vertex $v \in V(G)$, let O_v be the corresponding object in $\mathcal{O}$. For each O_v, let I_v be its projection onto the y-axis. Since O_v is path-connected, I_v is an interval, and let $p_v \geq q_v$ be its endpoints. Sort $\{p_v, q_v\}_{v \in V(G)}$ in $O(n \log n)$ time to obtain an ordering $z_1 \leq \cdots \leq z_{2n}$.

For each $i \in \mathbb{N}$, let $C_i = \{(x, y) \in \mathbb{R}^2 : (i-1)kd_{\min} \leq x < ikd_{\min}\}$ be the *i-th vertical strip*. For each $i \in \mathbb{N}$, the *i-th horizontal strip* R_i is defined as follows. If i is odd, then $R_i = \{(x, y) \in \mathbb{R}^2 : z_{(i+1)/2} \leq y \leq \frac{z_{(i+1)/2} + z_{(i+3)/2}}{2}\}$, whereas if i is even, then $R_i = \{(x, y) \in \mathbb{R}^2 : \frac{z_{i/2} + z_{(i+2)/2}}{2} \leq y \leq z_{(i+2)/2}\}$. Note that some horizontal strips might be degenerate, i.e., they are a horizontal line. See Figure 7.6 for an example.

We first construct a tree decomposition of G. Consider a path T with $4n-2$ vertices $\{t_1, \ldots, t_{4n-2}\}$ and, for each $1 \leq i \leq 4n-2$, let $X_{t_i} = \{v \in V(G) : O_v \cap R_i \neq \varnothing\}$. Clearly, for each $v \in V(G)$, there exists i with $v \in X_{t_i}$. Consider now an edge $uv \in E(G)$. There exists a point $(x, y) \in \mathbb{R}^2$ contained in both O_u and O_v, and this point belongs to some R_k. Therefore, $\{u, v\} \subseteq X_{t_k}$. Finally, for each $v \in V(G)$, the horizontal strips intersecting O_v are consecutive, and so $\{t_i \in V(T) : v \in X_{t_i}\}$ induces a subpath of T. This shows that $\mathcal{T} = (T, \{X_{t_i}\}_{t_i \in V(T)})$ is a tree decomposition of G and it is easy to see that such tree decomposition can be computed in $O(n)$ time.

We now construct a layering of G as follows. For each $j \in \mathbb{N}$, let V_j be the set of vertices whose corresponding objects have leftmost points inside C_j. It is easy to see that $(V_1, V_2, \ldots)$ is a partition of $V(G)$. Consider now i and j with $i - j \geq 2$. If $u \in V_i$ and $v \in V_j$, then O_v does not intersect C_i, as each object has diameter at most $k \cdot d_{\min}$, and O_u does not intersect C_{i-1}. Hence, O_u does not intersect O_v and so $uv \notin E(G)$. Therefore, $(V_1, V_2, \ldots)$ is a layering of G, which can clearly be computed in $O(n)$ time.

Consider an arbitrary bag X_{t_i} and layer V_j as defined above. Suppose that i is odd (the case i even is similar and thus left to the reader). Let $v \in X_{t_i} \cap V_j$. Then O_v intersects the line $y = z_{\frac{i+1}{2}}$, or else the lowermost points of O_v would have y-coordinate strictly between consecutive z_j's. Moreover, the leftmost points of O_v belong to C_j. But then, since the diameter of O_v is at most $k \cdot d_{\min}$, each object O_v with $v \in X_{t_i} \cap V_j$, intersects the line $y = z_{\frac{i+1}{2}}$ in a point with x-coordinate in the interval $[(j-1)kd_{\min}, (j+1)kd_{\min}]$. Consider now a family $\mathcal{F}$ of axis-aligned closed boxes with pairwise disjoint interiors and satisfying the following properties: Each box has size $d_{\min}/\sqrt{2}$, lower side on the line $y = z_{\frac{i+1}{2}}$, and the union of the lower sides of these boxes covers the horizontal line segment $\{(x, y) \in \mathbb{R}^2 : (j-1)kd_{\min} \leq x \leq (j+1)kd_{\min} \text{ and } y = z_{\frac{i+1}{2}}\}$. Clearly, there exists such a family $\mathcal{F}$ of size $\lceil 2\sqrt{2}k \rceil$. Since $\mathcal{O}$ is c-fat and each object in $\mathcal{O}$ has size at least $d_{\min}/\sqrt{2}$, we have that each box in $\mathcal{F}$ intersects at most c pairwise non-intersecting objects from $\mathcal{O}$. This implies that there are at most $\lceil 2\sqrt{2}k \rceil c$ pairwise non-intersecting objects from $\mathcal{O}$ whose corresponding vertices belong to $X_{t_i} \cap V_j$, thus concluding the proof. □

As mentioned above, the similarly-sized constraint cannot be dropped, for example because of the class of disk graphs. However, we will see in Section 7.5 that, for any fixed $d \geq 2$, the class of intersection graphs of c-fat families of objects in $\mathbb{R}^d$ is efficiently fractionally tree-α-fragile, a property weaker than boundedness of layered tree-independence number.

An important special case of Theorem 7.22 is that of unit disk graphs, for which we can obtain the following improved bound.

Theorem 7.23. *Let G be the intersection graph of a family $\mathcal{D}$ of n unit disks. It is possible to compute, in $O(n \log n)$ time, a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ and a layering $(V_1, V_2, \ldots)$ of G such that $|V(T)| = O(n)$ and, for each bag X_t and layer V_i, $\alpha(G[X_t \cap V_i]) \leq 3$. In particular, G has layered tree-independence number at most 3.*

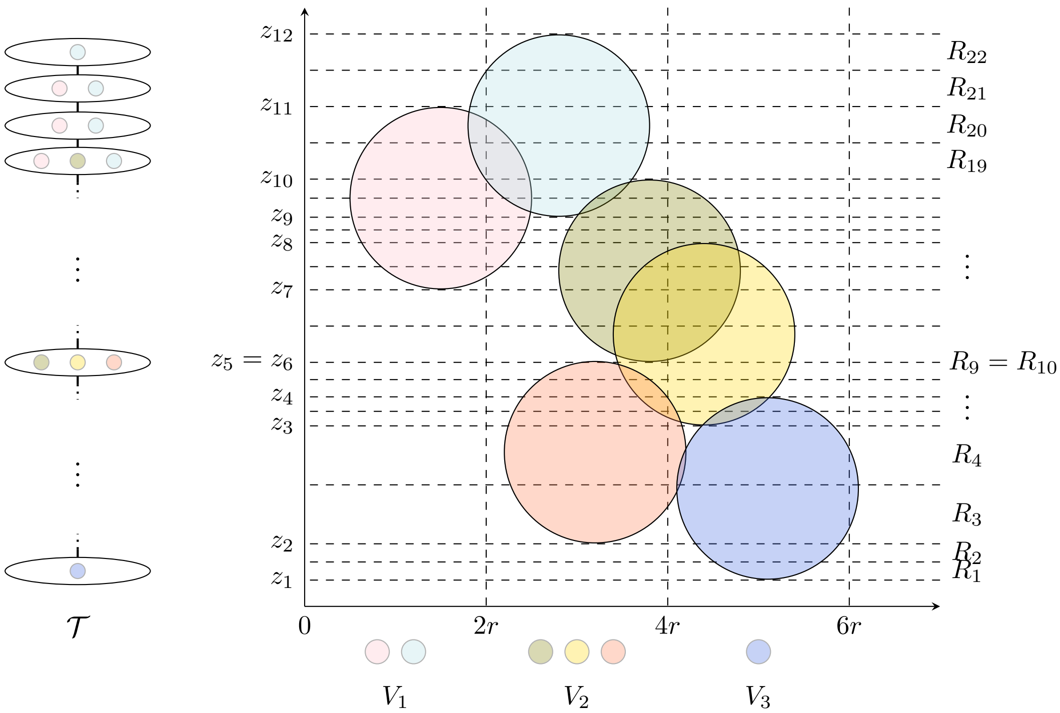

Figure 7.6: A tree decomposition and a layering of a unit disk graph witnessing layered tree-independence number at most 2.

Proof. Let r be the common radius of the disks. Without loss of generality, the family $\mathcal{D}$ is contained in the positive quadrant. For each vertex $v \in V(G)$, let D_v be the corresponding disk in $\mathcal{D}$. Let $z_1 \leq \cdots \leq z_{2n}$ be the ordering obtained as in the proof of Theorem 7.22 by projecting the disks onto the y-axis. The construction of a tree decomposition and layering of G is similar to that of Theorem 7.22. For each $i \in \mathbb{N}$, let $C_i = \{(x, y) \in \mathbb{R}^2 : 2(i-1)r \leq x < 2ir\}$ be the i-th vertical strip. For each $i \in \mathbb{N}$, the i-th horizontal strip R_i is defined as in Theorem 7.22. We now construct a tree decomposition $\mathcal{T} = (T, \{X_{t_i}\}_{t_i \in V(T)})$ and a layering $(V_1, V_2, \ldots)$ of G precisely as in Theorem 7.22 (see Figure 7.6).

Consider now an arbitrary bag X_{t_i} and layer V_j. Suppose that i is odd (the case i even is similar). Let $v \in X_{t_i} \cap V_j$. Then it is easy to see that the center of D_v must belong to the region $S = \{(x, y) \in \mathbb{R}^2 : 2ir - r \leq x < 2ir + r \text{ and } z_{\frac{i+1}{2}} - r \leq y \leq z_{\frac{i+1}{2}} + r\}$. But there are at most three non-intersecting unit disks with centers in S. Indeed, suppose four such centers lie inside S. By assumption, they are at pairwise distance greater than $2r$. If they are in convex position, then two of them must be at distance greater than $2\sqrt{2}r$. If they are not in convex position, then two of them must be at distance greater than $2\sqrt{3}r$. In either case we obtain a contradiction to the fact that the diameter of S is at most $2\sqrt{2}r$. □

Note that Theorem 7.23 (and hence Theorem 7.22 as well) cannot be extended to $\mathbb{R}^d$ with $d \geq 3$. Indeed, every 3-dimensional grid graph is the intersection graph of a family of unit balls in $\mathbb{R}^3$. Moreover, since the class of 3-dimensional grids has unbounded layered treewidth [62], it has unbounded layered tree-independence number as well. This follows from the fact that, by Ramsey's theorem, every graph with tree-independence number at most k and clique number at most p has treewidth at most $R(k+1, p+1) - 2$ [49].

It is easy to see that the collection of unit-width rectangles is not c-fat. However, we now argue that intersection graphs of unit-width rectangles form another class of bounded layered tree-independence number.

Theorem 7.24. *Let G be the intersection graph of a family $\mathcal{R}$ of n unit-width rectangles. It is possible to compute, in $O(n \log n)$ time, a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)}\})$ and a layering $(V_1, V_2, \ldots)$ of G such that $|V(T)| = O(n)$ and, for each bag X_t and layer V_i, $\alpha(G[X_t \cap V_i]) \leq 1$. In particular, G has layered tree-independence number at most 1.*

Proof. Let c be the common width of the rectangles. Without loss of generality, the family $\mathcal{R}$ is contained in the positive quadrant. For each vertex $v \in V(G)$, let R_v be the corresponding rectangle in $\mathcal{R}$. Let $z_1 \leq \cdots \leq z_{2n}$ be the ordering obtained as in the proof of Theorem 7.22 by projecting the rectangles onto the y-axis. The construction of a tree decomposition and layering of G is again similar to that of Theorem 7.22, the only difference being in the definition of the vertical strips: For each $i \in \mathbb{N}$, let $C_i = \{(x, y) \in \mathbb{R}^2 : (i-1)c \leq x < ic\}$ be the i-th vertical strip. We then construct a tree decomposition $\mathcal{T} = (T, \{X_{t_i}\}_{t_i \in V(T)})$ and a layering $(V_1, V_2, \ldots)$ of G precisely as in Theorem 7.22.

Consider an arbitrary bag X_{t_i} and layer V_j. Suppose that i is odd (the case i even is similar). Let $v \in X_{t_i} \cap V_j$. Then the rectangle R_v must intersect the line $y = z_{\frac{i+1}{2}}$ and the leftmost points of R_v belong to C_j. Clearly, there are no two disjoint such rectangles, thus concluding the proof. □

We now consider VPG and EPG graphs.

Theorem 7.25. *Let G be a graph on n vertices together with a grid representation $\mathcal{R} = (\mathcal{G}, \mathcal{P}, x)$ such that each path in $\mathcal{P}$ has horizontal part of length at most ℓ, for some fixed $\ell \geq 1$, and number of bends constant. It is possible to compute, in $O(n \log n)$ time, a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ and a layering $(V_1, V_2 \dots)$ of G such that $|V(T)| = O(n)$ and, for each bag X_t and layer V_i,*

- $\alpha(G[X_t \cap V_i]) \leq 2\ell$, *if* $x = v$;
- $\alpha(G[X_t \cap V_i]) \leq 6\ell - 1$, *if* $x = e$.

In particular, G has layered tree-independence number at most 2ℓ, if $x = v$, and at most $6\ell - 1$, if $x = e$.

Proof. For each vertex $v \in V(G)$, let P_v be the corresponding path in $\mathcal{P}$. Let $z_1 \leq \cdots \leq z_{2n}$ be the ordering obtained as in the proof of Theorem 7.22 by projecting the paths onto the y-axis. The construction of a tree decomposition and layering of G is again similar to that of Theorem 7.22, the only difference being in the definition of the vertical strips: For each $i \in \mathbb{N}$, let $C_i = \{(x, y) \in \mathbb{R}^2 : (i-1)\ell \leq x < i\ell\}$ be the i-th vertical strip. We then construct a tree decomposition $\mathcal{T} = (T, \{X_{t_i}\}_{t_i \in V(T)})$ and a layering $(V_1, V_2, \dots)$ of G precisely as in Theorem 7.22.

Consider an arbitrary bag X_{t_i} and layer V_j. Suppose that i is odd (the case i even is similar). Let I be an independent set of $G[X_{t_i} \cap V_j]$. Observe that, for each $v \in I \subseteq X_{t_i} \cap V_j$, the path P_v contains a grid-point at the intersection of the row of $\mathcal{G}$ indexed by $z_{\frac{i+1}{2}}$ and a column of $\mathcal{G}$ indexed by p, for some $(j-1)\ell \leq p < (j+1)\ell$. If $x = v$, then each such grid-point belongs to at most one P_v with $v \in I$, from which $|I| \leq 2\ell$. If $x = e$, each of the at most $6\ell - 1$ grid-edges containing such grid-points belongs to at most one P_v with $v \in I$ (or else two distinct paths P_u and P_v with $u, v \in I$ share a grid-edge), from which $|I| \leq 6\ell - 1$. □

Remark 7.26. Let G be the intersection graph of a family $\mathcal{O}$ of objects in $\mathbb{R}^2$. It is interesting to observe what are the key properties that guarantee boundedness of layered tree-independence number for G in the previous four results. What we need in our arguments are the following two properties:

- The objects in $\mathcal{O}$ have similar widths: the ratio of the largest ($w_{\max}$) and smallest ($w_{\min}$) object width (i.e., length of the horizontal part) is bounded;
- Any horizontal segment of length at most $w_{\max}$ stabs a bounded number of pairwise non-intersecting objects from $\mathcal{O}$.

We have observed earlier (in the proof of Lemma 7.14) that if a graph G has layered tree-independence number at most ℓ and the witnessing layering consists of c layers, then G has tree-independence number at most ℓc. We can apply this easy observation to the tree decomposition and layering built in the proof of Theorems 7.23 to 7.25, respectively, in order to obtain constant tree-independence number in case the corresponding geometric realization is contained in an axis-aligned rectangle with bounded width. This is the content of the next three results, whose proofs are easy and thus only sketched. These results will then be used in Section 7.6.4 to obtain simple Baker-style PTASes for Max Weight Independent Set with running times matching or improving the state of the art.

Corollary 7.27. *Let G be the intersection graph of a family $\mathcal{D}$ of n unit disks of common radius r such that its geometric realization is contained in an axis-aligned rectangle of width at most ℓ, for some integer $\ell > 0$. It is possible to compute, in $O(n \log n)$ time, a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of G such that $|V(T)| = O(n)$ and $\alpha(\mathcal{T}) \leq 3\lceil \frac{\ell}{2r} \rceil$.*

Proof. Build a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ and a layering $(V_1, V_2, \ldots)$ of G as in the proof of Theorem 7.23. Recall that, for each $j \in \mathbb{N}$, V_j is the set of vertices whose corresponding disks have leftmost points inside the j-th vertical strip $C_j = \{(x, y) \in \mathbb{R}^2 : 2(j-1)r \leq x < 2jr\}$. Since the geometric realization of G is contained in an axis-aligned rectangle of width at most ℓ, the disks in $\mathcal{D}$ intersect at most $\lceil \frac{\ell}{2r} \rceil$ vertical strips and so there are at most $\lceil \frac{\ell}{2r} \rceil$ layers. By Theorem 7.23, for each bag X_t and layer V_i, $\alpha(G[X_t \cap V_i]) \leq 3$. Therefore, $\alpha(\mathcal{T}) \leq \max_{t \in V(T)} \sum_{i \in \mathbb{N}} \alpha(G[X_t \cap V_i]) \leq 3\lceil \frac{\ell}{2r} \rceil$. □

Mutatis mutandis we obtain the following.

Corollary 7.28. *Let G be the intersection graph of a family $\mathcal{R}$ of n unit-width rectangles of common width c such that its geometric realization is contained in an axis-aligned rectangle of* *width at most ℓ, for some integer $\ell > 0$. It is possible to compute, in $O(n \log n)$ time, a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of G such that $|V(T)| = O(n)$ and $\alpha(\mathcal{T}) \leq \lceil \frac{\ell}{c} \rceil$.*

Corollary 7.29. *Let G be a n-vertex graph together with a grid representation $\mathcal{R} = (\mathcal{G}, \mathcal{P}, x)$ such that $\mathcal{G}$ contains at most ℓ columns, for some integer $\ell \geq 1$, and each path in $\mathcal{P}$ has number of bends constant. It is possible to compute, in $O(n \log n)$ time, a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of G such that $|V(T)| = O(n)$ and:*

- *$\alpha(\mathcal{T}) \leq \ell$, if $x = v$;*
- *$\alpha(\mathcal{T}) \leq 3\ell - 1$, if $x = e$.*

Proof. Note that a direct application of Theorem 7.25 gives the upper bounds 2ℓ (if $x = v$) and $6\ell - 1$ (if $x = e$). These can be easily improved as follows. Simply build a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ as in the proof of Theorem 7.25. Now, let $t \in V(T)$ and let I be an independent set of $G[X_t]$. For each $v \in I$, the path P_v contains a grid-point on a fixed row of $\mathcal{G}$. But these grid-points are within ℓ columns, and so $|I| \leq \ell$, if $x = v$, and $|I| \leq 3\ell - 1$, if $x = e$. □

In general, bounded layered tree-independence number implies sublinear (in the number of vertices) tree-independence number. More specifically, Theorems 7.22, 7.24 and 7.25, paired with Lemma 7.14, have the following immediate consequence.

Corollary 7.30. *If G is a graph on n vertices belonging to one of the following classes, then G has $O(\sqrt{n})$ tree-independence number:*

- *Intersection graphs of c-fat k-similarly-sized families of objects in $\mathbb{R}^2$, for some constants c and k (hence, in particular, unit disk graphs);*
- *Intersection graphs of unit-width rectangles in $\mathbb{R}^2$;*
- *VPG or EPG graphs where each path has bounded horizontal part and number of bends.*

In particular, unit disk graphs have $O(\sqrt{n})$ tree-independence number, from which it is easy to see that unit disk graphs of bounded degree have $O(\sqrt{n})$ treewidth. The latter was first observed by Fomin et al. [83].

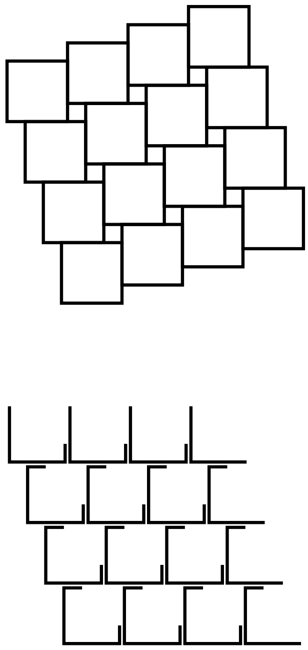

Figure 7.7: The 4×4-grid graph as the intersection graph of a family of unit squares, and as a VPG/EPG graph where each path has bounded horizontal part and at most 3 bends.

We now argue that the bounds obtained in Corollary 7.30 are tight up to constant factors. Since every grid graph can be realized as the intersection graph of a family of unit disks or unit-width rectangles, and is a VPG/EPG graph where each path has bounded horizontal part and number of bends (see Figures 7.5 and 7.7), it is enough to show that the $n \times n$-grid graph has tree-independence number $\Omega(n)$. In order to do so, we first recall some definitions. A pair of vertex subsets (A, B) is a *separation* in a graph G if $A \cup B = V(G)$ and there is no edge between $A \setminus B$ and $B \setminus A$. The *size* of a separation (A, B) is the quantity $|A \cap B|$. A separation (A, B) is *balanced* if $|A \setminus B| \leq 2|V(G)|/3$ and $|B \setminus A| \leq 2|V(G)|/3$. We also need the following known result (see [46, Lemma 7.20] for a proof).

Lemma 7.31 (Folklore)**.** *Let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ be a tree decomposition of a graph G. There exists a balanced separation (A, B) in G such that $A \cap B = X_t$, for some bag X_t of $\mathcal{T}$.*

Lemma 7.32. *The $n \times n$-grid graph has tree-independence number $\Omega(n)$.*

Proof. Let G be the $n \times n$-grid graph and let $\mathcal{T}$ be an arbitrary tree decomposition of G. It is well known (and an easy exercise) that, for $n \geq 4$, every balanced separation of G has size at least $n/4$. Lemma 7.31 then implies that there exists a bag of $\mathcal{T}$ of size at least $n/4$. But since G is bipartite, we obtain that $\alpha(\mathcal{T}) \geq n/8$. □

7.4 Fractional tree-α-fragility

In this section we start working with fractional tree-independence-number-fragility (tree-α-fragility for short), whose definition is stated in Chapter 3. Classes of bounded tree-independence number or efficiently fractionally tw-fragile classes are easily seen to be efficiently fractionally tree-α-fragile. Hence, the family of efficiently fractionally tree-α-fragile classes contains these two incomparable families (to see that they are incomparable, consider chordal graphs and planar graphs). We now identify one more subfamily, consisting of classes of bounded layered tree-independence number.

Lemma 7.33. *Let $\ell \in \mathbb{N}$ and let G be a graph. For each $r \in \mathbb{N}$, given a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ and a layering $(V_0, V_1, \ldots)$ of G such that, for each bag X_t and layer V_i, $\alpha(G[X_t \cap V_i]) \leq \ell$, it is possible to compute in $O(|V(G)|+|E(G)|+|V(G)|\cdot|V(T)|)$ time a $(1-1/r)$-general cover $\mathcal{C}$ of G and, for each $C \in \mathcal{C}$, a tree decomposition of $G[C]$ with independence number at most $\ell(r-1)$. In particular, if every graph in a class $\mathcal{G}$ has layered tree-independence number at most ℓ, then $\mathcal{G}$ is fractionally* tree-α*-fragile with $f(r) = \ell(r-1)$.*

Proof. Fix $r \in \mathbb{N}$. Let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ and $(V_0, V_1, \ldots)$ be the given tree decomposition and layering of G, respectively. For each $m \in \{0, \ldots, r-1\}$, let $C_m = \bigcup_{i \not\equiv m \pmod r} V_i$. We claim that $\mathcal{C} = \{C_m : 0 \leq m \leq r-1\}$ is a $(1-1/r)$-general cover of G with tree-independence number at most $\ell(r-1)$. Observe first that each $v \in V(G)$ is not covered by exactly one element of $\mathcal{C}$ and so it belongs to $r-1 = (1-1/r)|\mathcal{C}|$ elements of $\mathcal{C}$. Let now $C \in \mathcal{C}$. Each component K of $G[C]$ is contained within $r-1$ consecutive layers and so, since $\alpha(G[X_t \cap V_i]) \leq \ell$ for each bag X_t and layer V_i, restricting the bags in $\mathcal{T}$ to $V(K)$ gives a tree decomposition of K with independence number at most $\ell(r-1)$. We then merge the tree decompositions of the components of $G[C]$ into a tree decomposition of $G[C]$ with independence number at most $\ell(r-1)$. □

Remark 7.34. The argument used in Lemma 7.33 also implies that, if every graph in a class $\mathcal{G}$ has bounded layered treewidth, then $\mathcal{G}$ is fractionally tw-fragile.

But what are necessary conditions for fractional tree-α-fragility? Contrary to fractional tw-fragility, where sublinear-size separators are needed [67], one should expect that in the case of fractional tree-α-fragility it is not the size of a separator that has to be small but rather its independence number. In order to formally prove this, we introduce some notation.

Recall that a pair of vertex subsets (A, B) is a *separation* in a graph G if $A \cup B = V(G)$ and there is no edge between $A \setminus B$ and $B \setminus A$. The *separator* of a separation (A, B) is the set $A \cap B$. The *size* of a separation (A, B) is the quantity $|A \cap B|$, whereas the *independence number* of (A, B) is the quantity $\alpha(G[A \cap B])$. A separation (A, B) is *balanced* if $|A \setminus B| \leq 2|V(G)|/3$ and $|B \setminus A| \leq 2|V(G)|/3$. The minimum independence number of a balanced separation in G is denoted by $s\text{-}\alpha(G)$. For a graph class $\mathcal{G}$, let $s\text{-}\alpha_{\mathcal{G}}(n)$ denote the smallest non-negative integer such that every graph in $\mathcal{G}$ with at most n vertices has a balanced separation of independence number at most $s\text{-}\alpha_{\mathcal{G}}(n)$. In other words, $s\text{-}\alpha_{\mathcal{G}}(n) = \max\{s\text{-}\alpha(G) : G \in \mathcal{G}, |V(G)| \leq n\}$. We say that $\mathcal{G}$ has *separators of sublinear independence number* if $\lim_{n\to\infty} \frac{s\text{-}\alpha_{\mathcal{G}}(n)}{n} = 0$.

Lemma 7.35. *Let $\mathcal{G}$ be a fractionally* tree-*α-fragile class and let $c \in \mathbb{N}$ be arbitrary. Then there exists $k \in \mathbb{N}$ such that, for each $G \in \mathcal{G}$ with $|V(G)| \geq k$, $s\text{-}\alpha(G) < |V(G)|/c$. In particular, every fractionally* tree-*α-fragile class has separators of sublinear independence number.*

Proof. Since $\mathcal{G}$ is fractionally tree-α-fragile, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every $r \in \mathbb{N}$, every $G \in \mathcal{G}$ has a $(1-1/r)$-general cover $\mathcal{C}$ such that, for each $C \in \mathcal{C}$, tree-$\alpha(G[C]) \leq f(r)$. We show that the statement holds by taking $k = 3cf(2c)$. To this end, let $G \in \mathcal{G}$ be an arbitrary graph with $|V(G)| \geq k$, and let $\mathcal{C}$ be a $(1 - \frac{1}{2c})$-general cover of G as above. Then there exists $C \in \mathcal{C}$ such that $|C| \geq (1 - \frac{1}{2c})|V(G)|$. Moreover, there exists a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of $G[C]$ with independence number at most $f(2c)$. By Lemma 7.31, there exists a separation (A, B) in $G[C]$ such that $A \cap B = X_t$, for some bag X_t of $\mathcal{T}$, and $|A \setminus B| \leq 2|C|/3$ and $|B \setminus A| \leq 2|C|/3$. Observe now that $(A \cup (V(G) \setminus C), B \cup (V(G) \setminus C))$ is a balanced separation in G with separator $X = (A \cap B) \cup (V(G) \setminus C) = X_t \cup (V(G) \setminus C)$. Moreover,

$$\alpha(G[X]) \leq \alpha(G[X_t]) + |V(G) \setminus C| \leq f(2c) + \frac{1}{2c}|V(G)| \leq \frac{1}{3c}|V(G)| + \frac{1}{2c}|V(G)| < \frac{1}{c}|V(G)|.$$

This implies that $s\text{-}\alpha(G) < |V(G)|/c$, as claimed. □

Recall from Corollary 7.16 that large induced bicliques are an obstruction to small layered tree-independence number. Lemma 7.35 immediately implies that they remain an obstruction to fractional tree-α-fragility in hereditary graph classes. In fact, this is true even if the graph class is not hereditary, as we show next. Here the *induced biclique number* of a graph G is the maximum $n \in \mathbb{N}$ such that the complete bipartite graph $K_{n,n}$ is an induced subgraph of G.

Lemma 7.36. *Every fractionally* tree-α*-fragile graph class has bounded induced biclique number.*

Proof. Observe first that, if $\mathcal{C}$ is a β-general cover of a graph G and H is an induced subgraph of G, then $\mathcal{C} \cap H = \{C \cap V(H) : C \in \mathcal{C}\}$ is a β-general cover of H. Recalling that tree-independence number does not increase when taking induced subgraphs, it is therefore enough to show the following. For any function $f : \mathbb{N} \to \mathbb{N}$ and integer $r > 2$, there exists n such that no $(1 - 1/r)$-general cover of $K_{n,n}$ has tree-independence number less than $f(r)$. To this end, fix arbitrary $f : \mathbb{N} \to \mathbb{N}$ and $r > 2$. Consider a copy H of $K_{n,n}$, with $n > f(r)/(1-2/r)$. Let $\mathcal{C}$ be a $(1-1/r)$-general cover of H. Then every vertex of H belongs to at least $(1 - 1/r)|\mathcal{C}|$ elements of $\mathcal{C}$ and so there exists $C \in \mathcal{C}$ of size at least $2n(1 - 1/r)$. Let A and B be the two bipartition classes of H. Then $|A \cap C| \geq |C| - |B| \geq 2n(1 - 1/r) - n = n(1 - 2/r) > f(r)$ and, similarly, $|B \cap C| > f(r)$. Therefore, $H[C]$ contains $K_{f(r),f(r)}$ as an induced subgraph and since tree-$\alpha(K_{f(r),f(r)}) = f(r)$ [49], tree-$\alpha(H[C]) \geq f(r)$. □

Lemma 7.35 has another consequence. Recall that, for fixed $\alpha > 0$, a graph G is an α*-expander* if, for every $S \subseteq V(G)$ of size at most $|V(G)|/2$, there are at least $\alpha|S|$ edges of G with exactly one endpoint in S. 3-regular α-expanders on n vertices are known to exist for each sufficiently large even n and each of their separations has size $\Omega(n)$ (see, e.g., [61, 67]). Therefore, using [67, Lemma 28] to bound the size of a separation imply that 3-regular α-expanders are not fractionally tree-α-fragile and the same holds for 1-subdivisions of 3-regular α-expanders. We thus obtain the following result.

Corollary 7.37. *There exist classes of subcubic $K_{2,2}$-free graphs which are not fractionally* tree-α*-fragile.*

We conclude this section with another explicit construction of a graph class which is not fractionally tree-α-fragile. The *d-dimensional grid of side length n*, denoted $G_{d,n}$, is the graph with vertex set $[n]^d = \{(x_1, \ldots, x_d) : x_i \in \{1, 2, \ldots, n\}$ for each $i\}$, where two vertices $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ are adjacent if and only if $\sum_{1 \leq i \leq d} |x_i - y_i| = 1$.

Lemma 7.38. *Let $I, J \subseteq \mathbb{N}$ with I infinite and $J \neq \{1\}$. The class $\{G_{d,n} : d \in I, n \in J\}$ is not fractionally* tree-α*-fragile.*

Proof. Fix arbitrary $f\colon \mathbb{N} \to \mathbb{N}$ and $r > 2$. For such a choice, fix $d \in \mathbb{N}$ such that $\frac{r-4}{2r}d + 1 \geq R(3, f(r))$, where $R(3, f(r))$ denotes a Ramsey number. We now show that every $(1 - 1/r)$-general cover of $G_{d,n}$ has tree-independence at least $f(r)$. Let $\mathcal{C}$ be a $(1 - 1/r)$-general cover of $G_{d,n}$. Then every vertex of $G_{d,n}$ belongs to at least $(1 - 1/r)|\mathcal{C}|$ elements of $\mathcal{C}$ and so there exists $C \in \mathcal{C}$ containing at least $(1 - 1/r)|V(G_{d,n})| = (1 - 1/r)n^d$ vertices of $G_{d,n}$. Fix such a C and let G be the subgraph of $G_{d,n}$ induced by C. We claim that tree-$\alpha(G) \geq f(r)$.

Observe first that, for each $v \in V(G_{d,n})$, we have $d \leq d_{G_{d,n}}(v) \leq 2d$. Hence, $2|E(G_{d,n})| = \sum_{v \in V(G_{d,n})} d_{G_{d,n}}(v) \geq d \cdot n^d$. Consider now the graph G' obtained from $G_{d,n}$ by deleting the vertex set C. Clearly, G' has at most n^d/r vertices. Since deleting a vertex from $G_{d,n}$ decreases the number of edges of the resulting graph by at most $2d$, we have that $|E(G)| \geq |E(G_{d,n})| - 2d|V(G')|$, from which $\sum_{v \in V(G)} d_G(v) \geq d \cdot n^d - 2 \cdot 2d \cdot n^d/r = d \cdot n^d(1 - 4/r)$. Therefore, the average degree of G is at least $d(1 - 4/r)$ and so $\mathrm{tw}(G) \geq \frac{r-4}{2r}d$, for example by [37, Corollary 1]. This implies that every tree decomposition of G has a bag of size at least $\frac{r-4}{2r}d + 1 \geq R(3, f(r))$ and, since G is triangle-free, it follows that tree-$\alpha(G) \geq f(r)$. □

However, as we shall see in the next section, any family of bounded-dimensional grids is fractionally tree-α-fragile.

7.5 Intersection graphs of fat objects

Let $d \geq 2$ be an arbitrary but fixed integer. The main result of this section is the following: The class of intersection graphs of c-fat collections of objects in $\mathbb{R}^d$, defined in Section 7.2, is efficiently fractionally tree-α-fragile (Result C). More precisely, we show the following.

Theorem 7.39. *Let $\mathcal{O}$ be a c-fat collection of objects in $\mathbb{R}^d$ and let G be its intersection graph. For each $r_0 > 1$, let $f(r_0) = 2\left\lceil \frac{1}{1-\left(1-\frac{1}{r_0}\right)^{\frac{1}{d}}} \right\rceil$. Then we can compute in linear time a $(1-1/r_0)$-general cover $\mathcal{C}$ of G of size at most $(f(r_0)/2-1)^d$. Moreover, for each $C \in \mathcal{C}$, we can compute in linear time a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of $G[C]$, with $|V(T)| \leq |V(G)|+1$, such that $\alpha(\mathcal{T}) \leq cf(r_0)^{2d}$.*

Before proving Theorem 7.39, we outline the idea in the case of disk graphs in $\mathbb{R}^2$. Suppose first that we are trying to find a $(1-1/r_0)$-general cover $\mathcal{C}$ of bounded tree-independence number of a unit disk graph. We can build each element of the cover starting from an appropriate grid in the plane as follows. Suppose that $\mathcal{H}(y)$ is a grid in $\mathbb{R}^2$, indexed by some $y \in \mathbb{R}^2$, splitting the plane into a collection $\mathcal{B}$ of squares of side length $2r_0$. We first discard all disks intersecting $\mathcal{H}(y)$. The vertices corresponding to the remaining disks will form the element $C(y)$ of the cover. We can obtain a tree decomposition for the subgraph induced by $C(y)$ as follows. We add a node for each square $B \in \mathcal{B}$ and associate to this node a bag containing precisely the vertices whose corresponding disks lie in B. We then connect the nodes appropriately to obtain a tree. The resulting tree decomposition will have small independence number since, inside each square B, there are at most $4r_0^2$ pairwise non-intersecting disks. Shifting the grid $\mathcal{H}(y)$ around the plane via the vector y and proceeding as above will ensure that every vertex of the unit disk graph is covered by sufficiently many elements.

The situation is more challenging if disks have different radiuses. When both large and small disks occur, if the grids are too dense (i.e., they divide the plane into very small squares), then large disks will not belong to most elements of the cover, whereas if the grids are too sparse (i.e., they divide the plane into very large squares), then there might be too many pairwise non-intersecting small disks inside each square. To resolve this, we use an idea from [75, Theorem 4]. Specifically, we sort disks into different ranks according to their radius, so that the larger the radius the smaller the rank. Large disks will be "covered" by sparse grids, whereas small disks will be "covered" by dense grids. For each possible value i of the rank, we will consider grids of rank i arising in a quadtree-like manner from a fixed rank-0 grid (a sparesest grid), and we will discard rank-i disks intersecting rank-i grids. The vertices corresponding to the remaining disks will form an element $C(y)$ of the cover. We will then add a node for each square B_i induced by the rank-i grid, and associate to this node a bag containing precisely the vertices whose corresponding disks intersect B_i and have rank at most i. Finally, for each node t_i corresponding to a rank-i square B_i, we will add the edge t_it_j if t_j corresponds to the rank-j square B_j, with $j > i$, such that B_j is contained in B_i. As before, we will shift the grids around the plane via y to ensure that we obtain indeed a general cover.

Proof of Theorem 7.39. In this proof, $[n]$ denotes the set $\{0, 1, \ldots, n\}$. Fix an arbitrary $r_0 > 1$. In the following, for ease of notation, we simply let $r := f(r_0) = 2\left\lceil \frac{1}{1-\left(1-\frac{1}{r_0}\right)^{\frac{1}{d}}} \right\rceil$, and denote by O_v the object corresponding to the vertex $v \in V(G)$. By possibly rescaling, we may assume that each object in the collection $\mathcal{O}$ has size at most 1. For each $v \in V(G)$, define the *rank* of the object O_v as the quantity $\mathrm{rk}(O_v) = \lfloor \log_{\frac{1}{r}} s(O_v) \rfloor$. Let $k_0 = \max_{v \in V(G)} \mathrm{rk}(O_v)$. For each $0 \leq i \leq k_0$, $1 \leq j \leq d$ and $y = (y_1, \ldots, y_d) \in [\frac{r}{2} - 1]^d$, let $\mathcal{H}^i_j(y)$ be the set of points in $\mathbb{R}^d$ whose j-th coordinate is equal to $m_j (\frac{1}{r})^{i-1} + y_j \sum_{k=i}^{k_0+1} (\frac{1}{r})^k$, for some $m_j \in \mathbb{Z}$, and let $\mathcal{H}^i(y) = \bigcup_{1 \leq j \leq d} \mathcal{H}^i_j(y)$. Moreover, let $V^i = \{v \in V(G) : \mathrm{rk}(O_v) = i\}$, $C^i(y) = \{v \in V^i : O_v \cap \mathcal{H}^i(y) = \varnothing\}$, and $C(y) = \bigcup_{0 \leq i \leq k_0} C^i(y)$. See Figure 7.8.

Claim 7.40. *$\mathcal{C} = \{C(y) : y \in [\frac{r}{2} - 1]^d\}$ is a $(1 - 1/r_0)$-general cover of G of size $(f(r_0)/2 - 1)^d$.*

Proof. For each $1 \leq j \leq d$, let e_j be the unit vector in $\mathbb{R}^d$ whose j-th coordinate is 1. Let $v \in V(G)$ and let $i = \mathrm{rk}(O_v)$. Then $v \in C(y)$ for some $y \in [\frac{r}{2} - 1]^d$ if and only if O_v does not intersect $\mathcal{H}^i(y)$, and the latter happens if and only if O_v does not intersect $\mathcal{H}^i_j(y)$ for any $1 \leq j \leq d$. Note that $\mathcal{H}^i_j(y)$ is a collection of hyperplanes in $\mathbb{R}^d$, which are orthogonal to the j-th axis and at pairwise distance at least $(\frac{1}{r})^{i-1} = r(\frac{1}{r})^i$. Moreover, for any $1 \leq j \leq d$, $\mathcal{H}^i_j(y + e_j)$ can be obtained by shifting $\mathcal{H}^i_j(y)$ along the j-th axis of a quantity $\sum_{k=i}^{k_0+1} (\frac{1}{r})^k$, and it is easy to see that, since $r \geq 2$, we have $(\frac{1}{r})^i < \sum_{k=i}^{k_0+1} (\frac{1}{r})^k < 2(\frac{1}{r})^i$. On the other hand, since $\mathrm{rk}(O_v) = i$, it follows from the definition of rank that O_v can be enclosed in a box of size $(\frac{1}{r})^i$. We now count the number of points $y \in [\frac{r}{2} - 1]^d$ such that O_v does not intersect $\mathcal{H}^i(y)$. For fixed $1 \leq j \leq d$, the previous observations imply that there is at most one value of y_j for which a point $y \in [\frac{r}{2} - 1]^d$ is such that $\mathcal{H}^i_j(y) \cap O_v \neq \varnothing$, so at least $\frac{r}{2} - 1$ values of y_j for which y is such that $\mathcal{H}^i_j(y) \cap O_v = \varnothing$. Therefore, there are at least $(\frac{r}{2} - 1)^d$ points $y \in [\frac{r}{2} - 1]^d$ such that $\mathcal{H}^i(y)$ does not intersect O_v. Since the set $[\frac{r}{2} - 1]^d$ has size $(\frac{r}{2})^d$, the proportion of elements of $\mathcal{C}$ containing v is at least $\frac{(\frac{r}{2}-1)^d}{(\frac{r}{2})^d} = (1 - \frac{2}{r})^d \geq ((1 - \frac{1}{r_0})^{\frac{1}{d}})^d = 1 - \frac{1}{r_0}$, as claimed. ◇

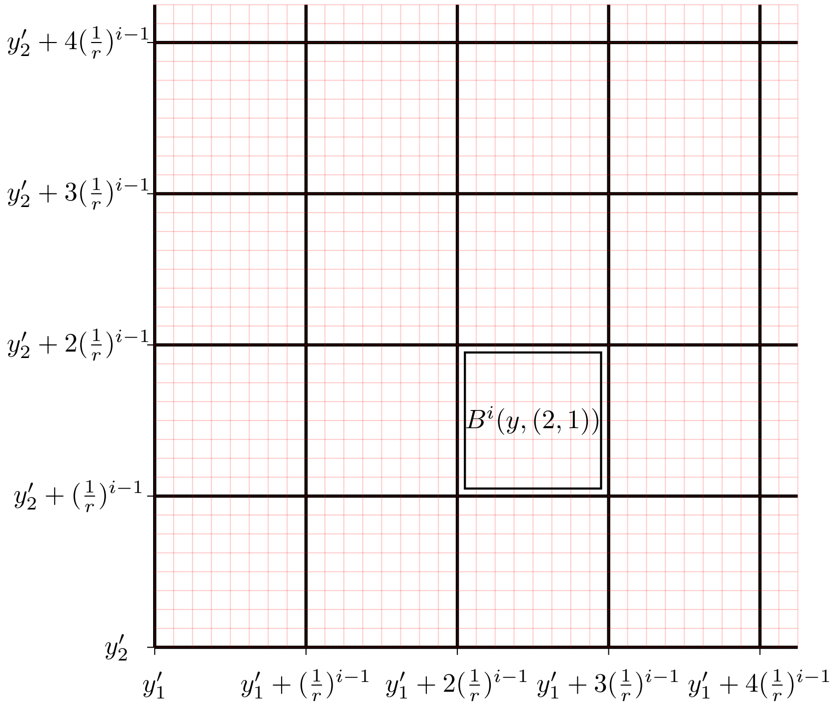

Figure 7.8: $\mathcal{H}^i(y)$ (black) and $\mathcal{H}^{i+1}(y)$ (red) in dimension 2, where $y'_j = y_j \sum_{k=i}^{k_0+1} (\frac{1}{r})^k$.

Note that, for fixed i and y as above, the collection of hyperplanes $\mathcal{H}^i(y)$ splits the space into boxes of size $(\frac{1}{r})^{i-1}$. We now consider these boxes. For a vector $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$, consider the box $B^i(y,m) = \{(x_1, \dots, x_d) \in \mathbb{R}^d : m_j(\frac{1}{r})^{i-1} + y_j \sum_{k=i}^{k_0+1} (\frac{1}{r})^k < x_j < (m_j + 1)(\frac{1}{r})^{i-1} + y_j \sum_{k=i}^{k_0+1} (\frac{1}{r})^k, \text{ for every } 1 \le j \le d\}$.

Claim 7.41. *For fixed $y \in [\frac{r}{2} - 1]^d$ and any $0 \le i' < i \le k_0$, $\mathcal{H}^{i'}(y) \subseteq \mathcal{H}^i(y)$. Moreover, for any $m \in \mathbb{Z}^d$, the box $B^i(y,m)$ is completely contained in a box of the form $B^{i'}(y,m')$ for exactly one vector $m' \in \mathbb{Z}^d$.*

Proof. Let $y = (y_1, \dots, y_d)$. Let $x = (x_1, \dots, x_d) \in \mathcal{H}^{i'}(y)$. There exists $1 \le j \le d$ such that $x \in \mathcal{H}_j^{i'}(y)$. Then there exists $m_j \in \mathbb{Z}$ such that

$$
\begin{aligned}
x_j = m_j \left(\frac{1}{r}\right)^{i'-1} + y_j \sum_{k=i'}^{k_0+1} \left(\frac{1}{r}\right)^k &= m_j \left(\frac{1}{r}\right)^{i'-i} \left(\frac{1}{r}\right)^{i-1} + y_j \sum_{k=i'}^{i-1} \left(\frac{1}{r}\right)^k + y_j \sum_{k=i}^{k_0+1} \left(\frac{1}{r}\right)^k \\
&= \left(m_j \left(\frac{1}{r}\right)^{i'-i} + y_j \sum_{k=i'}^{i-1} \left(\frac{1}{r}\right)^{k-(i-1)}\right) \left(\frac{1}{r}\right)^{i-1} + y_j \sum_{k=i}^{k_0+1} \left(\frac{1}{r}\right)^k.
\end{aligned}
$$

Since the coefficient of $(\frac{1}{r})^{i-1}$ is an integer, we conclude that $x \in \mathcal{H}_j^i(y)$ and so $\mathcal{H}^{i'}(y) \subseteq \mathcal{H}^i(y)$.

To prove the remaining statement simply recall that, for $m, m' \in \mathbb{Z}^d$, $B^i(y,m)$ is one of the boxes induced by $\mathbb{R}^d \setminus \mathcal{H}^i(y)$ and $B^{i'}(y,m')$ is one of the boxes induced by $\mathbb{R}^d \setminus \mathcal{H}^{i'}(y)$. Since $\mathcal{H}^i(y)$ is a refinement of $\mathcal{H}^{i'}(y)$, for any box of the form $B^i(y,m)$ there must be exactly one box of the form $B^{i'}(y,m)$ containing it. $\diamond$

We now construct a tree decomposition of $G[C(y)]$, for each element $C(y)$ of the $(1-1/r_0)$-general cover $\mathcal{C}$ defined above. Therefore, fix $y \in [\frac{r}{2}-1]^d$. For each $0 \leq i \leq k_0$ and $m \in \mathbb{Z}^d$, let $A^i(y,m) = \{v \in C^i(y) : O_v \cap B^i(y,m) \neq \varnothing\}$ and let $X_{t^i(y,m)} = \bigcup_{0 \leq k \leq i}\{v \in C^k(y) : O_v \cap B^i(y,m) \neq \varnothing\}$. In words, $X_{t^i(y,m)}$ is the set of vertices corresponding to objects of rank at most i in $C(y)$ and intersecting the box $B^i(y,m)$. For each pair (i,m), build a node $t^i(y,m)$ if $A^i(y,m) \neq \varnothing$ and associate to it the set $X_{t^i(y,m)}$, which will be the corresponding bag in the tree decomposition we are building. We say that $t^{i_1}(y,m_1)$ is a *parent* of $t^{i_2}(y,m_2)$ if the following conditions are satisfied: $i_1 < i_2$, $B^{i_1}(y,m_1) \supseteq B^{i_2}(y,m_2)$ and, among all pairs satisfying these two conditions, (i_1,m_1) has largest value of the first entry. Observe that, by Claim 7.41, each node $t^{i_2}(y,m_2)$ has at most one parent. For each pair of nodes $t^{i_1}(y,m_1)$, $t^{i_2}(y,m_2)$ such that $t^{i_1}(y,m_1)$ is a parent of $t^{i_2}(y,m_2)$, we then add the edge $t^{i_1}(y,m_1)t^{i_2}(y,m_2)$. We claim that the resulting graph $F(y)$ is acyclic. Suppose, to the contrary, that it contains a cycle with vertices $t^{i_0}(y,m_0), t^{i_1}(y,m_1), \dots, t^{i_{\ell-1}}(y,m_{\ell-1})$ in cyclic order. Without loss of generality, $t^{i_0}(y,m_0)$ is a parent of $t^{i_1}(y,m_1)$. This implies that, for each k, $t^{i_k}(y,m_k)$ is a parent of $t^{i_{k+1}}(y,m_{k+1})$ (indices modulo ℓ). Therefore, by definition of parent, $i_0 < i_1 < \cdots < i_{\ell-1}$ and $i_{\ell-1} < i_0$, a contradiction.

We then glue the components of $F(y)$ into a tree by adding a node t^{-1} and making t^{-1} adjacent to an arbitrary node of each component of $F(y)$. Let the resulting tree be $T(y)$. Observe that $|V(T(y))| \leq |V(G)|+1$. Indeed, $t^i(y,m)$ is a node of $T(y)$ only if $A^i(y,m) \neq \varnothing$ and, for fixed y, $A^{i_1}(y,m_1) \subseteq V(G)$ is disjoint from $A^{i_2}(y,m_2) \subseteq V(G)$. Let $\mathcal{T} = (T(y), \{X_{t^i(y,m)}\}_{t^i(y,m) \in V(T(y))})$, where we assign the empty bag to the node t^{-1}. Clearly, $\mathcal{T}$ can be computed in linear time.

Claim 7.42. $\mathcal{T} = (T(y), \{X_{t^i(y,m)}\}_{t^i(y,m) \in V(T(y))})$ *is a tree decomposition of* $G[C(y)]$.

Proof. We first check that (T1) holds. Let $v \in C(y)$ and suppose that $\mathrm{rk}(O_v) = i$. Then O_v intersects one of the boxes $B^i(y,m)$, for some $m \in \mathbb{Z}^d$, and so $v \in A^i(y,m)$. Therefore, $t^i(y,m) \in V(T(y))$ and $v \in X_{t^i(y,m)}$.

We now check that (T2) holds. Let $u, v \in C(y)$ such that $uv \in E(G)$. Then $O_u \cap O_v \neq \varnothing$ and let $x = (x_1, \dots, x_d)$ be a point in this intersection. Without loss of generality, $\mathrm{rk}(O_u) \leq \mathrm{rk}(O_v) = i$. Since $v \in C^i(y)$, $x \notin \mathcal{H}^i(y)$. This implies that x is contained in a box $B^i(y, m)$, for some $m \in \mathbb{Z}^d$, and so $v \in A^i(y, m)$, $t^i(y, m) \in V(T(y))$ and $\{u, v\} \subseteq X_{t^i(y,m)}$.

We finally check that (T3) holds. For $v \in C(y)$, let $T(y)_v$ be the subgraph of $T(y)$ induced by the set of nodes of $T(y)$ whose bag contains v. Let $v \in C(y)$ and suppose that $\mathrm{rk}(O_v) = i$. Observe first that there is a unique $m \in \mathbb{Z}^d$ such that $v \in X_{t^i(y,m)}$, or else $O_v \cap \mathcal{H}^i(y) \neq \varnothing$ and $v \notin C(y)$. Observe now that, by definition, $v \notin X_{t^{i_1}(y,m_1)}$ for any $i_1 < i$ and $m_1 \in \mathbb{Z}^d$. Suppose finally that $v \in X_{t^{i_1}(y,m_1)}$, for some $i_1 > i$ and $m_1 \in \mathbb{Z}^d$. Then O_v intersects $B^{i_1}(y, m_1)$ and, by Claim 7.41, there is a unique $m' \in \mathbb{Z}^d$ such that $B^{i_1}(y, m_1)$ is completely contained in $B^i(y, m')$ (it is easy to see that $m' = m$). Hence, $t^{i_1}(y, m_1)$ must have a parent, say $t^{i_2}(y, m_2)$ for some i_2 such that $i_1 > i_2 \geq i$ and $m_2 \in \mathbb{Z}^d$. This means that $B^{i_2}(y, m_2) \supseteq B^{i_1}(y, m_1)$, and so $v \in X_{t^{i_2}(y,m_2)}$. We then deduce inductively that there must be a path from $t^{i_1}(y, m_1)$ to $t^i(y, m)$ in $T(y)_v$. Therefore, $T(y)_v$ is connected. ◇

Claim 7.43. *$\alpha(\mathcal{T}) \leq cr^{2d}$.*

Proof. Fix an arbitrary node $t^i(y, m)$ of $T(y)$. We bound the independence number of the subgraph of $G[C(y)]$ induced by the bag $X_{t^i(y,m)}$. Observe first that, for any $v \in X_{t^i(y,m)}$, O_v intersects $B^i(y, m)$, which is a box of side length $(\frac{1}{r})^{i-1}$. Consider a collection $\mathcal{B}$ of r^{2d} generic closed boxes in $\mathbb{R}^d$ of side length $(\frac{1}{r})^{i+1}$ and such that their union is exactly $B^i(y, m)$. Let $P \subseteq X_{t^i(y,m)}$ be an independent set of $G[C(y)]$ and let $\mathcal{P} = \{O_v : v \in P\}$ be the corresponding subcollection of $\mathcal{O}$ of pairwise non-intersecting objects. For each $v \in P$, $\mathrm{rk}(O_v) \leq i$ and so $s(O_v) \geq (\frac{1}{r})^{i+1}$. Moreover, O_v intersects at least one box from $\mathcal{B}$. Therefore, since the collection $\mathcal{O}$ is c-fat, there are at most c objects in $\mathcal{P}$ intersecting any fixed box in $\mathcal{B}$, and so $|P| \leq cr^{2d}$, as claimed. ◇

This concludes the proof of Theorem 7.39. □

We note some consequences of Theorem 7.39. First, there exist fractionally tree-α-fragile classes of unbounded local tree-independence number.

Corollary 7.44. *The class of disk graphs is fractionally* tree-α*-fragile but has unbounded local tree-independence number.*

Proof. Let $\mathcal{G}$ be the class of intersection graphs of disks in $\mathbb{R}^2$. By Theorem 7.39, $\mathcal{G}$ is fractionally tree-α-fragile. Let now G_n be the graph obtained from the $n \times n$-grid graph by adding a dominating vertex. By the proof of Theorem 7.17, the class $\mathcal{G}' = \{G_n : n \in \mathbb{N}\}$ has unbounded local tree-independence number. We conclude by observing that $\mathcal{G}' \subseteq \mathcal{G}$. □

It is easy to see that every d-dimensional grid of side length n can be realized as the intersection graph of a family of unit balls in $\mathbb{R}^d$. Therefore, Theorem 7.39 and Lemma 7.38 immediately imply the following dichotomy.

Corollary 7.45. *Let $I, J \subseteq \mathbb{N}$, with $J \neq \{1\}$, and let $\mathcal{G} = \{G_{d,n} : d \in I, n \in J\}$ be a family of grids. Then $\mathcal{G}$ is fractionally* tree-α*-fragile if and only if I is finite.*

Note that Corollary 7.45 cannot be strengthened to bounded layered tree-independence number. Indeed, even though 2-dimensional grids have bounded layered tree-independence number (since, for example, they are planar), the family $\{G_{3,n} : n \in \mathbb{N}\}$ of 3-dimensional grids has unbounded layered tree-independence number, as observed in Section 7.3.1.

In Theorem 7.20 we showed that the class of graphs of bounded layered tree-independence number is closed under taking odd powers. It is not clear whether odd powers of graphs from *any* fractionally tree-α-fragile class form a fractionally tree-α-fragile class. Nevertheless, we now show that another fractionally tree-α-fragile class, namely that of intersection graphs of c-fat collections of objects in $\mathbb{R}^d$, is closed under taking odd powers.

Theorem 7.46. *Let G be the intersection graph of a c-fat collection of objects $\mathcal{O}$ in $\mathbb{R}^d$. Let k be a positive integer. Then G^{2k+1} is the intersection graph of a $(3^d(2k+1)^d c)$-fat collection of objects in $\mathbb{R}^d$. Moreover, given G and $\mathcal{O}$, such a collection can be computed in time polynomial in $|V(G)|$.*

Proof. For each $v \in V(G)$, denote by O_v the object in $\mathcal{O}$ corresponding to the vertex $v \in V(G)$. For each non-negative integer j, let $N^j[O_v]$ be the subset of objects from $\mathcal{O}$ corresponding to vertices in $N^j_G[v]$, and let $O^j_v = \bigcup_{O \in N^j[O_v]} O$. Note that O^j_v is path-connected and compact in $\mathbb{R}^d$.

Consider the collection of objects $\mathcal{O}^k = \{O^k_v : v \in V(G)\}$. We claim that its intersection graph G' is isomorphic to G^{2k+1}. The map sending each $v \in V(G)$ to O^k_v gives a natural bijection between the vertex sets of G^{2k+1} and G'. Consider now two distinct vertices $u, v \in G'$, corresponding to O^k_u and O^k_v, respectively. We have that u and v are adjacent in G' if and only if there exist $O_x \subseteq O^k_u$ and $O_y \subseteq O^k_v$ with $O_x \cap O_y \neq \varnothing$, for some $x, y \in V(G)$ (note that it might be $x = y$). But $d_G(u, x)$ and $d_G(v, y)$ are both at most k, from which $d_G(u, v) \leq 2k + 1$, and so u and v are adjacent in G^{2k+1}. Conversely, if u and v are distinct vertices adjacent in G^{2k+1}, then $d_G(u, v) \leq 2k + 1$. Take a shortest u, v-path P in G. Let P_u be the subpath of P with endpoint u and containing vertices at distance at most k from u. Similarly, let P_v be the subpath of P with endpoint v and containing vertices at distance at most k from v. Then $\bigcup_{x \in V(P_v)} O_x \subseteq O^k_v$, $\bigcup_{x \in V(P_u)} O_x \subseteq O^k_u$ and $\bigcup_{x \in V(P_v)} O_x \cap \bigcup_{x \in V(P_u)} O_x \neq \varnothing$, from which $O^k_u \cap O^k_v \neq \varnothing$.

We now show that the collection $\mathcal{O}^k$ is $(3^d(2k + 1)^d c)$-fat. Suppose, to the contrary, that for some r there exists a closed box R of side length r intersecting a subcollection $\mathcal{P} \subseteq \mathcal{O}^k$ of more than $3^d(2k + 1)^d c$ pairwise non-intersecting objects of size at least r. Let $\mathcal{P} = \{O^k_{v_1}, \ldots, O^k_{v_m}\}$, for some $m > 3^d(2k + 1)^d c$. Let R' be the box of side length $3r$ with the same center as R.

Claim 7.47. *For each $i \in \{1, \ldots, m\}$, there exists $A_i \in N^k[O_{v_i}]$ of size at least $\frac{r}{2k+1}$ and such that $A_i \cap R' \neq \varnothing$.*

Proof of Claim 7.47. Suppose, to the contrary, that there exists $i \in \{1, \ldots, m\}$ such that every object in $N^k[O_{v_i}]$ either does not intersect R' or has size less than $\frac{r}{2k+1}$. We distinguish two cases.

Case I: Every object in $N^k[O_{v_i}]$ intersects R'. By the assumption from the previous paragraph, every object in $N^k[O_{v_i}]$ has size less than $\frac{r}{2k+1}$. In particular, O_{v_i} is contained in a box of size less than $\frac{r}{2k+1}$. We show inductively that, for each $j \in \{0, \ldots, k\}$, $O^j_{v_i}$ is contained in a box of size less than $(2j+1)\frac{r}{2k+1}$. The base case $j = 0$ follows from the previous observation. Take now $j \in \{0, \ldots, k-1\}$ and suppose that $O^j_{v_i}$ is contained in a box X of size less than $(2j + 1)\frac{r}{2k+1}$. Since every object in $N^k[O_{v_i}]$ has size less than $\frac{r}{2k+1}$, every object in $N^{j+1}[O_{v_i}] \setminus N^j[O_{v_i}]$ is contained in a box of size less than $\frac{r}{2k+1}$, and moreover every such object must intersect the box

X. Therefore, $O_{v_i}^{j+1}$ is contained in a box of size less than $(2(j+1)+1)\frac{r}{2k+1}$. Taking $j = k$, we deduce that $O_{v_i}^k$ is contained in a box of size less than $(2k+1)\frac{r}{2k+1} = r$, contradicting the fact that $O_{v_i}^k$ has size at least r.

Case II: There exists $B \in N^k[O_{v_i}]$ not intersecting R'. Let $u \in V(G)$ be the vertex corresponding to B. Then, as $N^k[O_{v_i}]$ intersects R, there exists a path $u_1 \cdots u_p$ in G from $u_1 = u$ to some u_p of length at most $2k$ satisfying the following property: For each $j \in \{1, \ldots, p\}$, O_{u_j} belongs to $N^k[O_{v_i}]$ and O_{u_p} intersects R. Since B does not intersect R' and O_{u_p} intersects $R \subseteq R'$, there exists an index $\ell \in \{2, \ldots, p\}$ such that $O_{u_{\ell-1}}$ does not intersect R' whereas O_{u_j} intersects R' for all $j \geq \ell$. By assumption, each O_{u_j} with $j \geq \ell$ has size less than $\frac{r}{2k+1}$. Then, using a similar argument as in Case I, it is easy to see that $O = \bigcup_{j=\ell}^{p} O_{u_j}$ is contained in a box of size less than r. But O intersects R (as O_{u_p} does) and so O is entirely contained in R'. Since $O_{u_{\ell-1}}$ intersects O_{u_ℓ} and the latter is contained in R', we conclude that $O_{u_{\ell-1}}$ intersects R', a contradiction. $\diamond$

By the previous claim, for each $i \in \{1, \ldots, m\}$, $N^k[O_{v_i}]$ contains an object A_i of size at least $\frac{r}{2k+1}$ which intersects the box R' of side length $3r$. Since $\mathcal{P}$ consists of pairwise non-intersecting objects, the same holds for the family $\mathcal{A} = \{A_i : i \in \{1, \ldots, m\}\}$. Moreover, again by assumption, $|\mathcal{A}| > 3^d(2k+1)^d c$. Observe now that the box R' can be decomposed into a union of $3^d(2k+1)^d$ sub-boxes, each of side length $\frac{r}{2k+1}$. Then, by the pigeonhole principle, there must be at least $c+1$ distinct objects in $\mathcal{A}$ all intersecting the same sub-box of side length $\frac{r}{2k+1}$, contradicting the fact that $\mathcal{O}$ is c-fat. $\square$

We conclude this section by observing that even powers of fractionally tree-α-fragile classes need not be fractionally tree-α-fragile. Indeed, similarly to the proof of Lemma 7.21, and using Lemma 7.36, we obtain the following.

Lemma 7.48. *Fix an even $k \in \mathbb{N}$. The class $\mathcal{G}$ of chordal graphs is fractionally* tree-α*-fragile but the class $\{G^k : G \in \mathcal{G}\}$ is not.*

7.6 PTASes

In this section we prove Results A, B, D. Before providing the corresponding PTASes, we highlight some examples of problems captured by the framework of (c, h, ψ)-Max Weight Induced

Subgraph, which is addressed by Result A and which was defined in Section 7.1.1. To this purpose, it is useful to recall the following well-known observations (see, e.g., [9]): the subgraph and minor containment relations, as well as the property of being q-colorable, for fixed q, are all expressible in MSO_2. This immediately implies that problems such as Max Weight Independent Set, Max Weight Induced Forest and Max Weight Induced Planar Subgraph fall in the framework. The same holds for Max Weight Induced q-Colorable Subgraph, which is the problem that, for a fixed positive integer q and given a vertex-weighted graph G, asks to find a maximum-weight subset $F \subseteq V(G)$ such that $G[F]$ is q-colorable. The unweighted case $q = 2$ is known as Max Bipartite Subgraph [109].

Another example is the following. For a positive integer k, let $\mathcal{H}_k$ be a set of connected graphs which are contained in K_k. For fixed $\mathcal{H}_k$, Max $\mathcal{H}_k$-Free Node Set is the problem that, given a graph G, asks to find a maximum-size subset $F \subseteq V(G)$ such that $G[F]$ is $\mathcal{H}_k$-subgraph-free [121]. A notable special case is Max k-Dependent Set [57], the problem of finding a maximum-size induced subgraph of maximum degree at most k (the case $k = 1$ is also known as Dissociation Set [132, 152]). Max $\mathcal{H}_k$-Free Node Set corresponds to $(k-1, 1, \psi)$-Max Weight Induced Subgraph, where ψ is the MSO_2 formula expressing the property that none of the finitely many graphs in $\mathcal{H}_k$ is a subgraph of $G[F]$. We refer the reader to [84] for several other examples of problems which are special cases of (c, h, ψ)-Max Weight Induced Subgraph.

Result B and Result D concerns Max Weight Independent $\mathcal{H}$-Packing and its natural generalisation, Max Weight Distance-d $\mathcal{H}$-Packing, the former we defined in Section 5.1 with some examples. For fixed positive integers d and h, given a graph G and a finite family $\mathcal{H} = \{H_j\}_{j \in J}$ of connected non-null subgraphs of G such that $|V(H_j)| \leq h$ for every $j \in J$, a *distance-d $\mathcal{H}$-packing* in G is a subfamily $\mathcal{H}' = \{H_i\}_{i \in I}$ of subgraphs from $\mathcal{H}$ that are at pairwise distance at least d in G (that is, they share no common vertex and the length of the shortest path between them is at least d). If we are also given a weight function $w\colon J \to \mathbb{Q}_+$, Max Weight Distance-$d$ $\mathcal{H}$-Packing is the problem of finding a distance-d $\mathcal{H}$-packing in G of maximum weight. The case $d = 2$ coincides with Max Weight Independent $\mathcal{H}$-Packing.

7.6.1 Finding large induced sparse subgraphs satisfying a CMSO_2-definable near-monotone property in efficiently fractionally tree-α-fragile classes

In this section we show that (c, h, ψ)-Max Weight Induced Subgraph admits a PTAS on every efficiently fractionally tree-α-fragile class (Result A). The following result will be crucial for our proof.

Theorem 7.49 (Lima et al. [122]). *For every k, c and CMSO_2 formula ψ, there exists a positive integer $g(k, c, \psi)$ such that the following holds. Let G be a graph given along with a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ of G such that $\alpha(\mathcal{T}) \leq k$, and let $w\colon V(G) \to \mathbb{Q}_+$ be a weight function. Then, in time $g(k, c, \psi) \cdot |V(G)|^{O(R(k+1,c+1))} \cdot |V(T)|$, we can find a set $F \subseteq V(G)$ such that*

- *$G[F] \models \psi$,*
- *$\omega(G[F]) \leq c$,*
- *F is of maximum weight subject to the conditions above,*

or conclude that no such set exists.

Theorem 7.50. *Let $h \in \mathbb{N}$ and let ψ be a CMSO_2 formula expressing an h-near-monotone property, let $f\colon \mathbb{N} \to \mathbb{N}$ be a function and let $c \in \mathbb{N}$. There exists an algorithm that, given*

- *$r \in \mathbb{N}$ with $r > h$,*
- *a n-vertex graph G equipped with a $(1 - 1/r)$-general cover $\mathcal{C} = \{C_1, C_2, \ldots\}$ and, for each i, a tree decomposition $\mathcal{T}_i = (T_i, \{X_t\}_{t\in V(T_i)})$ of $G[C_i]$ with $\alpha(\mathcal{T}_i) \leq f(r)$,*
- *and a weight function $w\colon V(G) \to \mathbb{Q}_+$,*

in time $|\mathcal{C}| \cdot g(f(r), c, \psi) \cdot t \cdot n^{O(R(f(r)+1,c+1))}$, where $t = \max_i |V(T_i)|$ and g is the function from Theorem 7.49, either returns a subset $F \subseteq V(G)$ such that $G[F] \models \psi$, $\omega(G[F]) \leq c$, and $w(F)$ is at least a factor $(1 - h/r)$ of the optimal, or concludes that no such set F exists.

Proof. Observe first that, if an admissible solution F in G exists (i.e., $F \subseteq V(G)$ is such that $G[F] \models \psi$ and $\omega(G[F]) \le c$), then there exists a system $\{R_v \subseteq F : v \in F\}$ of subsets of F such that $F \setminus \bigcup_{v \in F \setminus C_i} R_v$ is an admissible solution in $G[C_i]$, for each $i \ge 1$. Indeed, since ψ expresses an h-near-monotone property, $G[F \setminus \bigcup_{v\in F\setminus C_i} R_v] \models \psi$. Moreover, $\omega(G[F \setminus \bigcup_{v\in F\setminus C_i} R_v]) \le \omega(G[F]) \le c$.

For each $i \ge 1$, we proceed as follows. Using the algorithm from Theorem 7.49, we simply look for optimal solutions $F_i \subseteq C_i$ in $G[C_i]$. For each i, finding F_i or concluding that no such set exists can be done in time $g(f(r), c, \psi) \cdot n^{O(R(f(r)+1,c+1))} \cdot t$. The total running time is then $|\mathcal{C}| \cdot g(f(r), c, \psi) \cdot t \cdot n^{O(R(f(r)+1,c+1))}$.

If, for some $i \ge 1$, there is no admissible solution in $G[C_i]$, then we return that no admissible solution exists in G. Correctness follows from the first paragraph. Otherwise, if optimal solutions exist in $G[C_i]$ for every i, then an optimal solution $Y \subseteq V(G)$ in G exists. Let now $\{R_v \subseteq Y : v \in Y\}$ be a system of subsets of Y as in the definition of h-near-monotonicity. We pick a C_i from $\mathcal{C}$ uniformly at random and observe that

$$\begin{aligned}
\mathbb{E}\left[w\left(Y \setminus \bigcup_{v\in Y\setminus C_i} R_v\right)\right] &= \mathbb{E}\left[\sum_{y\in Y} w(y)\mathbb{1}_{\{y\notin \bigcup_{v\in Y\setminus C_i} R_v\}}\right] \\
&= \sum_{y\in Y} w(y)\mathbb{P}\left(y \notin \bigcup_{v\in Y\setminus C_i} R_v\right) \\
&= \sum_{y\in Y} w(y)\left(1 - \mathbb{P}\left(y \in \bigcup_{v\in Y\setminus C_i} R_v\right)\right) \\
&= \sum_{y\in Y} w(y)\left(1 - \mathbb{P}\left(\bigcup_{v: y\in R_v} \{v \notin C_i\}\right)\right) \\
&\ge \sum_{y\in Y} w(y)\left(1 - \sum_{v: y\in R_v} \mathbb{P}(v \notin C_i)\right) \\
&\ge \sum_{y\in Y} w(y)\left(1 - \sum_{v: y\in R_v} \frac{1}{r}\right) \\
&\ge (1 - h/r)w(Y).
\end{aligned}$$

Hence, there exists an index j such that $w(Y \setminus \bigcup_{v\in Y\setminus C_j} R_v) \ge (1 - h/r)w(Y)$. We then return the subset F_j computed above. Since $Y \setminus \bigcup_{v\in Y\setminus C_j} R_v$ is an admissible solution in $G[C_j]$, the optimality of F_j in $G[C_j]$ implies that $w(F_j) \ge w(Y \setminus \bigcup_{v\in Y\setminus C_j} R_v) \ge (1 - h/r)w(Y)$. □

We remark that the near-monotonicity requirement in Theorem 7.50 is necessary. Indeed, for fixed r, it is not difficult to see that inducing an r-regular subgraph is a property expressible in CMSO_2 but which is not near-monotone for $r \geq 2$. On the other hand, for each fixed $r \in \{3, 4, 5\}$, given a planar graph G, the problem of finding a maximum-size subset of $V(G)$ inducing an r-regular subgraph does not admit a PTAS, unless $\mathsf{P} = \mathsf{NP}$ [5]. Moreover, boundedness of clique number of the solution is necessary as well. Indeed, being a clique is a monotone property that can be easily expressed by a CMSO_2 formula. On the other hand, the problem of finding a maximum-size clique on unit ball graphs in $\mathbb{R}^4$ does not admit a PTAS, unless the Exponential Time Hypothesis (ETH) fails [20].

7.6.2 Packing subgraphs at distance at least 2 in efficiently fractionally tree-α-fragile classes

In this section we show that Max Weight Independent $\mathcal{H}$-Packing admits a PTAS on every efficiently fractionally tree-α-fragile class (Result B). Such a PTAS relies on the following result.

Theorem 7.51 (Dallard et al. [49])**.** *Let k and h be two positive integers. Given a graph G and a finite family $\mathcal{H} = \{H_j\}_{j \in J}$ of connected non-null subgraphs of G such that $|V(H_j)| \leq h$ for every $j \in J$,* Max Weight Independent $\mathcal{H}$-Packing *can be solved in time $O(|V(G)|^{h(k+1)} \cdot |V(T)|)$ if G is given together with a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ with $\alpha(\mathcal{T}) \leq k$.*

Theorem 7.52. *Let $h \in \mathbb{N}$ and let $f \colon \mathbb{N} \to \mathbb{N}$ be a function. There exists an algorithm that, given*

- *$r \in \mathbb{N}$ with $r > h$,*
- *a n-vertex graph G equipped with a $(1 - 1/r)$-general cover $\mathcal{C} = \{C_1, C_2, \ldots\}$ and, for each i, a tree decomposition $\mathcal{T}_i = (T_i, \{X_t\}_{t \in V(T_i)})$ of $G[C_i]$ with $\alpha(\mathcal{T}_i) \leq f(r)$,*
- *a finite family $\mathcal{H} = \{H_j\}_{j \in J}$ of connected non-null subgraphs of G such that $|V(H_j)| \leq h$ for every $j \in J$,*
- *and a weight function $w \colon J \to \mathbb{Q}_+$ on the subgraphs in $\mathcal{H}$,*

returns in time $|\mathcal{C}| \cdot O(n^{h(f(r)+1)} \cdot t)$, where $t = \max_i |V(T_i)|$, an independent $\mathcal{H}$-packing in G of weight at least a factor $(1 - h/r)$ of the optimal.

The proof of Theorem 7.52 is similar to that of Theorem 7.50.

Proof. For each $i \geq 1$, we proceed as follows. Using the algorithm from Theorem 7.51, we simply compute a maximum-weight independent $\mathcal{H}$-packing $\mathcal{P}_i$ in $G[C_i]$ in time $O(n^{h(f(r)+1)} \cdot t)$. The total running time is then $|\mathcal{C}| \cdot O(n^{h(f(r)+1)} \cdot t)$. For a collection $\mathcal{A}$ of subgraphs of G, each isomorphic to a member of $\mathcal{H}$, and a subset $C \subseteq V(G)$, let $w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(A)$ and let $\mathcal{A} \cap C = \{A \in \mathcal{A} : A \subseteq C\}$. Observe that, given a subgraph H of G, each vertex $v \in V(H)$ is not contained in at most $|\mathcal{C}|/r$ elements of the $(1 - 1/r)$-general cover $\mathcal{C}$. Hence, $V(H)$ is contained in at least $(1 - |V(H)|/r)|\mathcal{C}|$ elements of $\mathcal{C}$. Let $\mathcal{P} = \{P_1, P_2, \ldots\}$ be an independent $\mathcal{H}$-packing in G of maximum weight. Then

$$\begin{aligned}
\sum_{C_i \in \mathcal{C}} w(\mathcal{P} \cap C_i) &= \sum_{C_i \in \mathcal{C}} \sum_{P_j \in \mathcal{P}} w(P_j) \mathbb{1}_{\{P_j \subseteq C_i\}} \\
&= \sum_{P_j \in \mathcal{P}} w(P_j) \sum_{C_i \in \mathcal{C}} \mathbb{1}_{\{P_j \subseteq C_i\}} \\
&\geq \sum_{P_j \in \mathcal{P}} w(P_j)(1 - |V(P_j)|/r)|\mathcal{C}| \\
&\geq \sum_{P_j \in \mathcal{P}} w(P_j)(1 - h/r)|\mathcal{C}| \\
&= |\mathcal{C}|(1 - h/r)w(\mathcal{P}).
\end{aligned}$$

By the pigeonhole principle, there exists $C_i \in \mathcal{C}$ such that $w(\mathcal{P} \cap C_i) \geq (1 - h/r)w(\mathcal{P})$. We then return the maximum-weight independent $\mathcal{H}$-packing $\mathcal{P}_i$ in $G[C_i]$ computed above. Since $\mathcal{P} \cap C_i$ is an independent $\mathcal{H}$-packing in $G[C_i]$, we have that $w(\mathcal{P}_i) \geq w(\mathcal{P} \cap C_i) \geq (1 - h/r)w(\mathcal{P})$. □

For the special case of Max Weight Independent Set on intersection graphs of c-fat collections of objects in a fixed d-dimensional space, we obtain the following.

Corollary 7.53. *There exists an algorithm that, given $r \in \mathbb{N}$, a c-fat collection $\mathcal{O}$ of n objects in $\mathbb{R}^d$ and its intersection graph G, and a weight function $w\colon V(G) \to \mathbb{Q}_+$, returns in time $(f(r)/2 - 1)^d \cdot O(n^{cf(r)^{2d}+2})$, where $f(r) = 2\left\lceil \frac{1}{1-\left(1-\frac{1}{r}\right)^{\frac{1}{d}}} \right\rceil$, an independent set in G of weight at least a factor $(1 - 1/r)$ of the optimal.*

Proof. Given $r \in \mathbb{N}$, we use Theorem 7.39 to compute in $O(n)$ time a $(1-1/r)$-general cover $\mathcal{C}$ of G of size at most $(f(r)/2-1)^d$. Moreover, for each $C \in \mathcal{C}$, we compute in $O(n)$ time a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ of $G[C]$, with $|V(T)| \leq n+1$, such that $\alpha(\mathcal{T}) \leq cf(r)^{2d}$. We finally apply the algorithm from Theorem 7.52 (with $h = 1$). The total running time is $(f(r)/2-1)^d \cdot O(n^{cf(r)^{2d}+2})$. □

7.6.3 Packing subgraphs at distance at least d in graphs with bounded layered tree-independence number or in intersection graphs of c-fat collections

In this section we prove Result D. The following result shows that, for each even $d \in \mathbb{N}$, Max Weight Distance-d $\mathcal{H}$-Packing admits a PTAS on every class of bounded layered tree-independence number.

Theorem 7.54. *Let $h, \ell \in \mathbb{N}$. Let d be an even positive integer. There exists an algorithm that, given*

- *$r \in \mathbb{N}$ with $r > h$,*
- *a n-vertex graph G equipped with a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)}\})$ and a layering $(V_1, V_2, \ldots)$ of G such that, for each bag X_t and layer V_i, $\alpha(G[X_t \cap V_i]) \leq \ell$,*
- *a finite family $\mathcal{H} = \{H_j\}_{j\in J}$ of connected non-null subgraphs of G such that $|V(H_j)| \leq h$ for every $j \in J$,*
- *and a weight function $w\colon J \to \mathbb{Q}_+$,*

returns, in time $r \cdot |V(T)| \cdot n^{O(r)}$, a distance-$d$ $\mathcal{H}$-packing in G of weight at least a factor $(1-h/r)$ of the optimal.

Proof. Let $d = 2k$. As observed in [122, Observation 3.9], for $I \subseteq J$, the subfamily $\mathcal{H}' = \{H_i\}_{i\in I}$ is a distance-d $\mathcal{H}$-packing in G if and only if $\mathcal{H}'$ is an independent $\mathcal{H}$-packing in the graph G^{d-1}. Therefore, using BFS, we first compute in $O(n^3)$ time the graph G^{2k-1}. Using the algorithm from Theorem 7.20, we compute in $O(|V(T)| \cdot n^2)$ time a tree decomposition $\mathcal{T}' = (T, \{X'_t\}_{t\in V(T)})$ of G^{2k-1} and a layering $(V'_1, \ldots, V'_{\lceil \frac{m}{2k-1} \rceil})$ of G^{2k-1} such that, for each bag X'_t and layer V'_i, $\alpha(G^{2k-1}[X'_t \cap V'_i]) \leq (4k-3)\ell$. Using the algorithm from Lemma 7.33, we compute in $O(n^2 + n \cdot |V(T)|)$ time a $(1-1/r)$-general cover $\mathcal{C}$ of G^{2k-1} of size r and, for each $C \in \mathcal{C}$, a tree decomposition of $G^{2k-1}[C]$ with independence number at most $\ell(4k-3)(r-1)$. Finally, we apply the approximation algorithm from Theorem 7.52 to obtain, in time $r \cdot O(n^{h(\ell(4k-3)(r-1)+1)} \cdot |V(T)|)$, an independent $\mathcal{H}$-packing in G^{2k-1} of weight at least a factor $(1-h/r)$ of the optimal. □

In a similar way, combining Theorem 7.52 with Theorems 7.39 and 7.46, we immediately obtain the following:

Theorem 7.55. *Let $d \in \mathbb{N}$ be even.* Max Weight Distance-d $\mathcal{H}$-Packing *admits a* PTAS *for intersection graphs of c-fat collections of objects in $\mathbb{R}^k$, for any fixed c and k.*

We remark that Theorem 7.54 cannot be extended to odd values of d, unless $\mathsf{P} = \mathsf{NP}$. Indeed, Eto et al. [80] showed that, for each $\varepsilon > 0$ and fixed odd $d \geq 3$, it is NP-hard to approximate Distance-d Independent Set to within a factor of $n^{1/2-\varepsilon}$ for chordal graphs. However, it is not clear whether Theorem 7.55 can be extended to odd values of d.

7.6.4 Packing independent unit disks, unit-width rectangles and paths with bounded horizontal part on a grid

We have seen in Section 7.3.1 that classes of intersection graphs of unit disks, unit-width rectangles and paths with bounded horizontal part on a grid have bounded layered tree-independence number and hence Max Weight Independent Set admits a PTAS when restricted to any of them thanks to either Theorem 7.50 or Theorem 7.52. In this section we provide PTASes with improved running time for Max Weight Independent Set on these three classes.

We begin with VPG/EPG graphs. The following result generalises the PTAS for Max Weight Independent Set on B_1-EPG graphs with bounded horizontal part given in [15, Theorem 6].

Theorem 7.56. *Let $c \in \mathbb{N}$.* Max Weight Independent Set *admits a PTAS when restricted to n-vertex graphs with a grid representation $\mathcal{R} = (\mathcal{G}, \mathcal{P}, x)$ such that:*

1. *each path in $\mathcal{P}$ has number of bends constant;*

2. the horizontal part of each path in $\mathcal{P}$ has length at most c.

If $x = v$, the running time is $O(c\lceil\frac{1}{\varepsilon}\rceil \cdot n^{\lceil\frac{1}{\varepsilon}\rceil c+3})$. If $x = e$, the running time is $O(c\lceil\frac{1}{\varepsilon}\rceil \cdot n^{3\lceil\frac{1}{\varepsilon}\rceil c+2})$.

Proof. Let G be a n-vertex graph with a grid representation $\mathcal{R} = (\mathcal{G}, \mathcal{P}, x)$ satisfying the conditions above. Without loss of generality, we may assume that all the paths in $\mathcal{P}$ contain only grid-points with non-negative coordinates. Moreover, we may assume that G is connected. Therefore, no column in $\mathcal{G}$ is unused and so $\mathcal{G}$ has at most $(c+1)n$ columns. Further note that since any path $P \in \mathcal{P}$ has number of bends constant, we can compute the horizontal part $h(P)$ of P in $O(1)$ time. Given $0 < \varepsilon < 1$, we fix $k = \lceil 1/\varepsilon \rceil$.

For any $i \geq 0$, we denote by X_i the set of vertices whose corresponding path contains a grid-edge $[(i,j),(i+1,j)]$ for some $j \geq 0$ (here and in the following $[(i,j),(i+1,j)]$ denotes the grid-edge with endpoints (i,j) and $(i+1,j)$). Note that we can compute the at most $(c+1)n-1$ non-empty sets $X_i's$ in $O(n)$ time. In view of applying a shifting argument, we now partition G into slices via the following. For any $d \in \{0,\dots,kc-1\}$, let $V_d = \bigcup_{\ell \in \mathbb{N}_0} X_{d+\ell kc}$ be the set of vertices whose corresponding path contains a grid-edge $[(d+\ell kc, j),(d+\ell kc+1, j)]$ for some $\ell, j \in \mathbb{N}_0$. We claim that, for any $d \in \{0,\dots,kc-1\}$, $G - V_d$ is disconnected. Indeed, after deleting V_d, no vertex whose horizontal part is contained in the interval $[0, d+\ell kc]$ can be adjacent to a vertex whose horizontal part is contained in the interval $[d+\ell kc+1, (c+1)n]$. Similarly, every component of $G - V_d$ admits a grid representation in which the number of columns is bounded by kc. By Corollary 7.29 and Theorem 7.51, for each component of $G - V_d$, we compute a maximum-weight independent set in $O(n^{kc+2})$ time, if $x = v$, or in $O(n^{3kc+1})$ time, if $x = e$. The union U_d of these independent sets over the components of $G - V_d$ is then an independent set of G and, after repeating the procedure above for each $d \in \{0,\dots,kc-1\}$, we return the maximum-weight set U among the U_d's. The total running time is then $O(kc \cdot n^{kc+3})$, if $x = v$, or $O(kc \cdot n^{3kc+2})$, if $x = e$.

It remains to show that $w(U) \geq (1-\varepsilon)w(\mathsf{OPT})$, where OPT denotes an optimal solution of Max Weight Independent Set with instance G. Note that, for any $d \in \{0,\dots,kc-1\}$, $\mathsf{OPT} \cap V_d$ is the set of vertices in OPT whose corresponding path contains a grid-edge $[(d+\ell kc, j),(d+\ell kc+1, j)]$ for some $\ell, j \in \mathbb{N}_0$. Since the horizontal part of each path has length at most c, we have that every vertex in OPT belongs to at most c distinct V_d's. Therefore, denoting by d_0 an index attaining $\min_{d\in\{0,\ldots,kc-1\}} w(\mathsf{OPT} \cap V_d)$, we have that

$$kc \cdot w(\mathsf{OPT} \cap V_{d_0}) \leq \sum_{d=0}^{kc-1} w(\mathsf{OPT} \cap V_d) \leq c \cdot w(\mathsf{OPT})$$

and so

$$w(\mathsf{OPT}) = w(\mathsf{OPT} \setminus V_{d_0}) + w(\mathsf{OPT} \cap V_{d_0}) \leq w(U) + \varepsilon \cdot w(\mathsf{OPT}),$$

thus concluding the proof. □

We note two consequences of Theorem 7.56. It provides a PTAS for Max Weight Independent Set on equilateral B_1-VPG graphs (i.e., B_1-VPG graphs where, for each path, its horizontal and vertical segment have the same length) where paths have bounded horizontal part and on unit B_k-VPG graphs (i.e., B_k-VPG graphs where each segment has unit length). Lahiri et al. [117] provided a $O(\log d)$-approximation algorithm for the unweighted version on equilateral B_1-VPG graphs, where d denotes the ratio between the maximum and minimum length of segments of paths, which for constant d gives a constant-factor approximation algorithm. Theorem 7.56 improves this to a PTAS. Theorem 7.56 also complements the $O(k^4)$-approximation algorithm for Dominating Set on unit B_k-VPG graphs provided by Chakraborty et al. [34].

We now pass to the PTASes for intersection graphs of unit disks and unit-width rectangles. The running time of our PTAS for unit disk graphs improves on the $(1 - 1/k)$-approximation algorithm with running time $O(kn^{4\lceil \frac{2(k-1)}{\sqrt{3}} \rceil})$ by Matsui [128].

Theorem 7.57. Max Weight Independent Set *admits a PTAS when restricted to:*

- *Intersection graphs of a family $\mathcal{D}$ of n unit disks of common radius* 1*. The running time is $O(\lceil \frac{3}{\varepsilon} \rceil \cdot n^{3\lceil \frac{3-\varepsilon}{2\varepsilon} \rceil + 3})$.*
- *Intersection graphs of a family $\mathcal{R}$ of n unit-width rectangles of common width* 1*. The running time is $O(\lceil \frac{2}{\varepsilon} \rceil \cdot n^{\lceil \frac{2}{\varepsilon} \rceil + 2})$.*

Proof. Since the PTASes are similar, we introduce some common notation. Let G be the intersection graph of the family $\mathcal{O}$, where $\mathcal{O}$ is either $\mathcal{D}$ or $\mathcal{R}$. Without loss of generality, we may assume that all objects in $\mathcal{O}$ are contained in the positive quadrant. Moreover, we may assume that G is connected. Therefore, $\mathcal{O}$ is contained in a grid $\mathcal{G}$ with $O(n)$ columns. For each $O \in \mathcal{O}$, we compute the horizontal part $h(O)$ of O (i.e., the projection of O onto the x-axis) in $O(1)$ time. Given $0 < \varepsilon < 1$, we fix $k = \lceil 2/\varepsilon \rceil$, in the case of rectangles, and $k = \lceil 3/\varepsilon \rceil$, in the case of disks. Let X_i be the set of vertices whose corresponding objects have horizontal part intersecting the half-open segment $[(i,0),(i+1,0))$. We can compute the X_i's in $O(n)$ time. We now partition G into slices as follows. For any $d \in \{0,\dots,k-1\}$, let $V_d = \bigcup_{\ell \in \mathbb{N}_0} X_{d+\ell k}$. As in the proof of Theorem 7.56, it is easy to see that, for any $d \in \{0,\dots,k-1\}$, each component of $G - V_d$ admits a geometric realization which is contained in an axis-aligned rectangle with width at most $k-1$. If the family consists of disks, for each component of $G - V_d$, we compute a maximum-weight independent set in $O(n^{3\lceil \frac{k-1}{2} \rceil + 2})$ time (thanks to Corollary 7.27 and Theorem 7.51). If the family consists of rectangles, for each component of $G - V_d$, we compute a maximum-weight independent set in $O(n^{k+1})$ time (thanks to Corollary 7.28 and Theorem 7.51). In either case, the union U_d of these independent sets over the components of $G - V_d$ is an independent set of G and, after repeating the procedure above for each $d \in \{0,\dots,k-1\}$, we return the maximum-weight set U among the U_d's. The total running time is then $O(k \cdot n^{3\lceil \frac{k-1}{2} \rceil + 3})$, in the case of disks, and $O(k \cdot n^{k+2})$, in the case of rectangles. Similarly to Theorem 7.56, it is easy to see that $w(U) \geq (1-\varepsilon)w(\mathsf{OPT})$, where OPT denotes an optimal solution of Max Weight Independent Set with instance G. □

7.7 Subexponential-time algorithms

Although the focus of this thesis is on polynomial-time algorithms, in this short section we note some consequences of our work in relation to subexponential-time algorithms. In particular, the observations from Section 7.3 immediately lead to subexponential-time algorithms for Max Weight Distance-d $\mathcal{H}$-Packing, for each fixed even $d \in \mathbb{N}$, on classes of bounded layered tree-independence number, as we argue below. We have listed in Section 7.6 some examples of problems falling in this framework. Further well-known examples of dual problems (minimization problems) are k-Separator (also known as k-Component Order Connectivity in its unweighted version) [11, 118] and its special case Min Weight 3-Path Vertex Cover [29, 118].

Corollary 7.58. *Let $\ell, d \in \mathbb{N}$ be fixed constants, with d even. Let G be a n-vertex graph for which we can compute, in time* $\mathsf{poly}(n)$*, a tree decomposition and a layering witnessing layered tree-independence number at most ℓ. Then* Max Weight Distance-d $\mathcal{H}$-Packing *can be solved in $2^{O(\sqrt{n}\log n)}$ time*[11]*.*

Proof. We first compute, in time $\mathsf{poly}(n)$, a tree decomposition and a layering witnessing layered tree-independence number at most ℓ. We then apply Lemma 7.14 and compute a tree decomposition of G with independence number $O(\sqrt{n})$. It is then enough to recall that Max Weight Distance-d $\mathcal{H}$-Packing is solvable in time $n^{O(k)}$, where k is the tree-independence number of the input graph, provided a tree decomposition with independence number at most k is given in input [122]. □

Remark 7.59. Note that, in Corollary 7.58, we require a tree decomposition and a layering giving constant layered tree-independence number. This is because we cannot directly use $O(\sqrt{n})$ tree-independence number in conjunction with the XP-approximation algorithm for tree-independence number of Dallard et al. [50] (for fixed k, there exists an algorithm that, given a n-vertex graph G, in time $2^{O(k^2)}n^{O(k)}$, either decides that tree-$\alpha(G) > k$, or outputs a tree decomposition of G with independence number at most $8k$), as this would not grant subexponential time.

Corollary 7.58 has interesting consequences. In particular, paired with Theorem 7.22, it immediately gives a $2^{O(\sqrt{n}\log n)}$-time algorithm for Max Weight Distance-d $\mathcal{H}$-Packing on intersection graphs of similarly-sized c-fat families of objects in $\mathbb{R}^2$. A $2^{O(\sqrt{n}\log n)}$-time algorithm for the special case $d = 2$ (i.e., for Max Weight Independent Set) on unit disk graphs was first given in [120]. More recently, de Berg et al. [53] provided $2^{O(\sqrt{n})}$-time algorithms for the *unweighted* version of many problems on intersection graphs of similarly-sized globally fat objects in $\mathbb{R}^d$ and this is tight under the ETH. Examples of such problems are Independent Set, Induced Matching, Distance-d Dominating Set and, more generally, problems whose solutions (or the complements thereof) can contain at most a constant number of vertices from any clique.

[11] An alternative way to obtain subexponential-time algorithms for some of the *unweighted* problems captured by Max Weight Distance-d $\mathcal{H}$-Packing follows from a general result of Korhonen and Lokshtanov [113], who provided $2^{O_H(n^{2/3}\log n)}$-algorithms on H-induced-minor-free graphs for unweighted problems such as Independent Set and Induced Matching. This result applies to every class of bounded layered tree-independence number, as this must be $K_{n,n}$-induced-minor-free, for some n.

Although the $2^{O(\sqrt{n})}$-time algorithm for Independent Set from [53] can be extended to handle the weighted version (see, e.g., [52]), to the best of our knowledge very little is known about the weighted case in general. The only lower bound we are aware of is by de Berg and Kisfaludi-Bak [52], hinting at the fact that the weighted versions could be considerably harder: Min Weight Dominating Set cannot be solved in $2^{o(n)}$ time on unit ball graphs in $\mathbb{R}^3$, unless the ETH fails. In fact, de Berg and Kisfaludi-Bak [52] asked to determine the complexity of the weighted versions of problems falling in the framework of [53] when restricted to intersection graphs of similarly-sized fat objects in $\mathbb{R}^2$. Corollary 7.58 partially answers this question by showing that the weighted versions of some of these problems, such as Max Weight Independent Set on the intersection graph of fat objects, admit subexponential-time exact algorithms, at the cost of an extra $\log n$ factor in the exponent. It is then natural to ask whether this $\log n$ factor can be shaved off or not. In other words, do the complexities of the weighted and unweighted versions match in $\mathbb{R}^2$?

Finally, it would be interesting to investigate whether the notion of fractional tree-α-fragility could be useful in the design of subexponential-time algorithms.

7.8 Concluding remarks and open problems

In this chapter we began investigating the notion of fractional tree-α-fragility, which allows to unify classes of sparse graphs (e.g., proper minor-closed) and dense graphs (e.g., geometric intersection graphs), and demonstrated its usefulness in the design of PTASes for maximization problems. In particular, we obtained an approximation meta-theorem for the broad family of fractionally tree-α-fragile classes. Besides generality, we believe that an additional strength of this approach lies in the simplicity of the approximation schemes obtained. We now list some open questions whose answers allow to identify the applicability limits of our frameworks.

The first natural direction for a possible extension is given by the following.

Open Problem 9. *Is it true that, for each even $d \in \mathbb{N}$,* Max Weight Distance-d $\mathcal{H}$-Packing *admits a PTAS on every efficiently fractionally* tree-α*-fragile class?*

Recall that, for each odd $d \geq 3$, we cannot hope for a positive answer to the previous question, unless $\mathsf{P} = \mathsf{NP}$ [80]. However, it is not clear whether the following question has still a negative answer.

Open Problem 10. *Let $d \in \mathbb{N}$ be odd. Does* Max Weight Distance-d $\mathcal{H}$-Packing *admit a* PTAS *for intersection graphs of c-fat collections of objects in $\mathbb{R}^k$, for fixed c and k?*

We observed in Lemma 7.35 that every fractionally tree-α-fragile class has separators of sublinear independence number. The converse does not hold in general, as shown by the following example. Let G_n be the complete bipartite graph $K_{n,f(n)}$ for some integer-valued function f of order $o(n)$. The class of graphs $\{G_n\}$ has separators of sublinear independence number but has unbounded induced biclique number and hence is not fractionally tree-α-fragile thanks to Lemma 7.36. However, the following might be true.

Open Problem 11. *Is it true that every hereditary class with separators of sublinear independence number is fractionally* tree-α*-fragile?*

We remark that the similar question of whether every subgraph-closed class with strongly sublinear separators is fractionally tw-fragile was asked by Dvořák [67] and is still open. It turns out that the answer is positive under the additional assumption of bounded maximum degree (see [67, Lemma 19]).

A *clique-based separator* of a graph G is a collection $\mathcal{S}$ of vertex-disjoint cliques whose union is a balanced separator of G. The *weight* of $\mathcal{S}$ is the quantity $\sum_{C \in \mathcal{S}} \log(|C|+1)$. De Berg et al. [53, 54] showed that several intersection graphs of geometric objects in the plane (e.g., map graphs and intersection graphs of convex globally fat objects, similarly-sized globally fat objects, or pseudo-disks) admit clique-based separators of weight $O(\sqrt{n})$ and used this to obtain algorithms with running time $2^{O(\sqrt{n})}$ for many problems on such graphs, as remarked in Section 7.7. Clearly, if a graph class $\mathcal{G}$ admits clique-based separators of sublinear weight, then it also admits separators of sublinear independence number. However, the class $\mathcal{G}'$ defined above shows that the converse does not hold, as each clique-based separator of a graph in $\mathcal{G}'$ has weight $\Omega(n)$.

Open Problem 12. *Is it true that every class of bounded layered tree-independence number admits clique-based separators of sublinear weight?*

It would be interesting to further investigate the notions of layered treewidth, layered tree-independence number and fractional tree-α-fragility for classes of string graphs and intersection graphs of pseudo-disks. Recall that a string graph is the intersection graph of a set of curves in the plane, where it can be assumed that no three curves meet at a single point (see, e.g., [6]). For an integer $k \geq 2$, if each curve is in at most k intersections with other curves, then the corresponding string graph is called a *k-string graph.* A *(g, k)-string graph* is defined analogously for curves on a surface of Euler genus at most g. In general, the class of string graphs is not fractionally tree-α-fragile (see Lemma 7.36). However, Dujmović et al. [64] showed that the class of (g, k)-string graphs has bounded layered treewidth. A natural question is what happens for its superclass[12] of string graphs of bounded maximum degree:

Open Problem 13. *Does the class of string graphs of bounded maximum degree have bounded layered treewidth?*

More generally, one can consider $K_{t,t}$-subgraph-free string graphs. Thanks to a result of Lee [119], these graphs have separators of size $O(\sqrt{n})$. However, the layered tree-independence number is unbounded, as is easily seen by considering 2-dimensional grids with a dominating vertex.

Open Problem 14. *Let $t \geq 2$. Is the class of $K_{t,t}$-subgraph-free string graphs fractionally* tree-α*-fragile? More generally, is the class of $K_{t,t}$-free string graphs fractionally* tree-α*-fragile?*

A set of objects in the plane is a collection of *pseudo-disks* if each object is bounded by a Jordan curve and, for each pair of objects, their boundaries intersect at most twice. De Berg et al. [54] showed that any intersection graph of pseudo-disks has a clique-based separator of weight $O(n^{2/3} \log n)$. Corollary 7.44 implies that the class of intersection graphs of pseudo-disks has unbounded layered tree-independence number. However, we ask the following.

Open Problem 15. *Is the class of intersection graphs of pseudo-disks fractionally* tree-α*-fragile?*

[12]The maximum degree of a k-string graph might be much less than k.

Chapter 8

Conclusion

In this thesis we contributed to the long-standing research on graph width parameters, both from a structural and algorithmic perspective. From a structural perspective, we classified (un)boundedness of width for several graph classes and width parameters; we also studied the equivalence between different width parameters when restricting to specific graph classes. From an algorithmic perspective, with the aid of width parameters, we provided polynomial-time exact algorithms for otherwise NP-hard problems when restricting to some specific graph classes, or polynomial-time approximation schemes when such problems remain NP-hard for the restricted graph classes.

We have been particularly interested in two graph width parameters introduced relatively recently, namely mim-width and sim-width. A plethora of algorithmic applications of boundedness of mim-width are known (see, e.g., [13]) and our interest in mim-width has been structural. In Chapter 4, we contributed to the open problems from [26] of classifying the (un)boundedness of mim-width of (H_1, H_2)-free graphs when H_1 is edgeless (i.e., $H_1 = rP_1$ for some $r \in \mathbb{N}$) or complete (i.e., $H_1 = K_r$ for some $r \in \mathbb{N}$). We obtained a complete dichotomy in the former case (for $r \geq 3$) and an almost-complete dichotomy in the latter case (for $r \geq 4$). Naturally, we think that the most interesting open problem in this respect is to obtain a complete dichotomy when H_1 is a complete graph on at least 4 vertices. The remaining open cases are stated in Open Problem 2. In Chapter 5, we provided the first algorithmic application of sim-width, namely we showed that List (d, k)-Colouring is in XP parameterized by the sim-width of a given branch decomposition of the input graph, thus answering a question from [111]. The most important and well-known open problem related to sim-width is whether Max Weight Independent Set is in XP parameterized by the sim-width of a given branch decomposition of the input graph. We showed that, if this is the case, then the meta-problem Max Weight Independent $\mathcal{H}$-Packing admits the same behaviour. Some recent progress concerning Max Weight Independent Set has been made by Bergougnoux et al. [14], who provided a quasipolynomial-time algorithm for graphs of bounded sim-width.

In Chapter 6, we considered how the relationship between non-equivalent width parameters changes once we restrict to some special graph class. More precisely, we provided a complete comparison of six graph width parameters (treewidth, clique-width, twin-width, mim-width, sim-width and tree-independence number) when restricted to $K_{t,t}$-subgraph-free graphs and line graphs, extending well-known results of Gurski and Wanke [95, 96], and an almost-complete comparison when restricted to the common superclass (for $t \geq 3$) of $K_{t,t}$-free graphs. The only open case left to obtain a complete dichotomy is stated in Open Problem 4, which asks whether tree-independence number dominates sim-width for the class of $K_{t,t}$-free graphs. Although this question remains open, some progress has been recently made by Abrishami et al. [2], who showed that tree-independence number dominates induced matching treewidth (a width parameter weaker than sim-width) for the class of $K_{t,t}$-free graphs.

In Chapter 7, we introduced layered tree-independence number and fractional tree-α-fragility. We showed that these notions are particularly useful for designing polynomial-time approximation schemes for geometric intersection graphs. One of our main contributions was proving that the class of intersection graphs of c-fat objects in any fixed dimension is fractionally tree-α-fragile, and hence admits polynomial-time approximation schemes for the NP-hard meta-problems (c, h, ψ)-Max Weight Induced Subgraph and Max Weight Independent Packing. If we further restrict the objects to be similarly-sized objects in $\mathbb{R}^2$, we additionally obtain boundedness of layered tree-independence number, which in turn gives polynomial-time approximation schemes with faster running time. Our structural understanding of fractional tree-α-fragility is still somewhat limited and we believe that it would be interesting to further investigate sufficient structural properties guaranteeing fractional tree-α-fragility. For example, in Open Problem 11, we asked whether hereditarily having separators of sublinear independence number grants fractional tree-α-fragility.

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