Semiclassical charge transfer in gravitational encounters

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Semiclassical charge transfer is considered in the context of gravitational encounters. A Schwarzschild radius is invoked as a constant of integration. The point of closest approach is greater than the positive Schwarzschild radius. Charge transfer and Newtonian gravity are considered in an acausal context. In the limit in which the Coulomb potential dominates, the Schrödinger time-dependent equation is regained.

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It is said [1] that quantum mechanics and general relatively are incompatible. Moreover black holes are discussed in their entirety on Google, Wikipedia. Popular reading includes 'A brief history of time' [2] and 'Quantum Gravity' [3]. However let us consider symmetric resonance charge transfer and/or elastic scattering where the projectile (P) has a large mass of M_P and a charge of Z_P whose centres coincide and the target (T) has a large mass of M_T and a charge of Z_T, whose centres again coincide. The electron is initially bound to the target and the impact velocity of the projectile is $\boldsymbol{v}$, where $v << 1$.

Consider the generalized Kohn Lagrangian (density) [4, 5]

$$L = \Psi^* \left(H - E\right) \Psi + \lambda \left[\frac{1}{1+f}\dot{R}^2 + R^2\dot{\Theta}^2 - \left(1+f\right)c^2\right] \Psi^* \Psi \tag{1}$$

where E is the total constant energy eigenvalue which may be interpreted as a Langrangian multiplier associated with the normalization of the wave function Ψ (see eqn.(19) below) and where λ is a Lagrange multiplier, f is given by

$$-1 < f(R) = -\frac{2m}{R} < 0 \tag{2}$$

where $2m$ is the freely adjustable constant of integration and the possibly negative Schwarzschild radius [6] in which case we have a bound state, and where azimuthal symmetry is assumed. For $m > 0$ the heavy-particle collision occurs in the collision plane. The $(\cdot)$ means time differentiation and

$$\mathbf{R} \equiv (R, \Theta) = \vec{TP} \tag{3}$$

where Θ is the polar angle with $\boldsymbol{v}$ as polar axis. The Hamiltonian is given by

$$H = -\frac{1}{2}\nabla^2_{\mathbf{r}} - \frac{1}{2\mu}\nabla^2_{\mathbf{R}} + \frac{(e^2 Z_P Z_T - G M_P M_T)}{R} - \frac{Z_T e^2}{r_T} - \frac{Z_P e^2}{r_P} \tag{4}$$

including the Newtonian-gravity potential which is the limit of general-relativity, and where $\mathbf{r}_T$, $\mathbf{r}_P$ and $\mathbf{r}$, are the position vectors of the electron relative to respectively the target and the projectile and their midpoint and G is the universal gravitational constant. The gravitational and electromagnetic potentials in eqn.(4) are treated on an equal footing. This is equivalent to adding both the electromagnetic and gravitational forces. The reduced mass μ is $M_P M_T / m_e (M_P + M_T)$ and the energy E is $\frac{1}{2\mu}k_0^2$ where $\mathbf{k}_0 = \mu\mathbf{v}$, neglecting the internal quantum energy of $-\frac{1}{2}Z_T^2$. We normalize (2) to be consistent with (4) so that

$$\frac{-2m\mu^2 c^2}{R} = -2\mu\left(\frac{e^2 Z_P Z_T - G M_P M_T}{R}\right). \tag{5}$$

Thus we have an effective Schwarzschild radius given by

$$2m = -2\frac{m_e^2}{c^2}\overline{\overline{G}}\left(\frac{M_P}{m_e} + \frac{M_T}{m_e}\right) + \frac{2e^2 Z_P Z_T}{c^2\mu} > 0 \tag{6}$$

where c is the speed of light in vacuum, e and m_e are the charge and mass respectively of the electron [7] and $\overline{\overline{G}}$ is given by

$$\overline{\overline{G}} = 6.673 \times 5.273^{-2} \text{ x } 10^{-42} \approx 0.24 \text{ x } 10^{-42}. \tag{7}$$

Clearly M_P and M_T must be exceedingly large, of the order of 10^{42} m_e for the electromagnetic and Newtonian gravitational potentials/forces to be commensurable. Clearly m must be non-negative for a collision, negative for a bound state. In order to consider symmetric resonance and/or elastic scattering ($\alpha > 0$ (eqn. 13)), for simplicity, we assume identical "particles" so that

$$Z_P = Z_T \quad \text{and} \quad M_P = M_T. \tag{8}$$

We apply an Euler-Jacobi second-order variational principle [4] to the Lagrangian of eqn.(1) so that we have [8] an elastic cross section

$$Q^{00} = 2\pi a_0^2 \int_0^\infty \rho d\rho \left[2\sin^2\eta_\rho^+ + 2\sin^2\eta_\rho^- - \sin^2\left(\eta_\rho^+ - \eta_\rho^-\right)\right] \tag{9}$$

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and a symmetric charge transfer cross section

$$Q^{01} = 2\pi a_0^2 \int_0^\infty \rho d\rho \sin^2\left(\eta_\rho^+ - \eta_\rho^-\right). \tag{10}$$

Here the impact parameter ρ is related semiclassically to the azimuthal quantum number by

$$\rho k_0 = l + \frac{1}{2} \tag{11}$$

and $\eta_\rho^\pm$ are the gerade/ungerade phase shifts, given semiclassically in the LCAO (linear combination of atomic orbitals) approximation by

$$\eta_\rho^\pm = \lim_{R\to\infty}\left[\rho\pi + \alpha\ln(2k_0R) - \arg\Gamma(\rho k_0 + \frac{1}{2} + i\alpha) + \int_{R_0}^{R} d\tilde{R}\left\{\left(k_0^2 - 2\mu\left(V_{00} \pm V_{01}\right) - \frac{\rho^2 k_o^2}{\tilde{R}^2}\right)^{1/2} - k_0\right\}\right] \tag{12}$$

where R_0 is the classical turning point and

$$\alpha = \left(Z_P Z_T e^2 - G M_P M_T\right)/(2v^2) > 0 \tag{13}$$

$$V_{00}(R) = \frac{\alpha}{R} \tag{14}$$

$$V_{01}(R) = 2Z_T\left(1 + Z_T R\right)e^{-Z_T R} \tag{15}$$

the latter being for the 1s-1s transition. It follows semiclassically (in two senses) and rationalising the numerator that

$$\eta_\rho^+ - \eta_\rho^- \approx -2\mu \int_{R_0}^\infty \frac{V_{01}(R)\,dR}{\sqrt{k_0^2 - 2\mu V_{00}(R) - \frac{\rho^2 k_0^2}{R^2}}} \tag{16}$$

since (15) is negligible compared to (14). Since we have $v << 1$ and are dealing with the ultimate symmetric resonant noncrossing collision [9] we can safely ignore any Stokes constant effects in the Stueckelberg oscillations. We may also neglect electron translation factors ($\exp\left(\pm i\mathbf{v}\cdot\mathbf{r}/2\right)$) and special relativity effects such as time dilation and mass contraction ($\sqrt{1 - v^2/c^2} \approx 1$). In addition the variational principle gives us the equations on page 106 of [6] except that his energy-integral equation (10.27) is amended to

$$\frac{1}{1+f}\dot{R}^2 + R^2\dot{\Theta}^2 - (1+f)\,c^2 = v^2 - c^2 \tag{17}$$

the difference being the v^2 on the right hand side, which comes from the $\frac{1}{2\mu}k_0^2$ energy of the relative motion of the heavy particles. Solving (17) for $\dot{R}$ we see that the classical relativistic amendment to equation (16) is given by

$$\eta_\rho^+ - \eta_\rho^- \approx -2\mu \int_{max(R_0,2m)}^\infty \frac{V_{01}(R)dR}{\sqrt{1+f\left(R\right)}\sqrt{k_0^2 - 2\mu V_{00}(R) - \frac{\rho^2 k_0^2}{R^2}}}. \tag{18}$$

The square root singularity is removable by making the change of variable $x^2 = R - 2m$.

Thus the expressions (10) and (18) remain causal whereas expressions (9) and (12) amended as in (18) will result in acausality [10].

In retrospect we come to the conclusion that consistency requires the Schrödinger time-independent equation in the modified form:

$$\left[-\frac{1}{2m_e}\nabla_\mathbf{r}^2 - \frac{1}{2\mu(1+f)}\nabla_\mathbf{R}^2 + \frac{(e^2 Z_P Z_T - G M_P M_T)}{R} - \frac{Z_T e^2}{r_T} - \frac{Z_P e^2}{r_P} - E\right]\Psi = 0 \tag{19}$$

to be consistent with equations (17) and (18). The effective heavy-particle reduced mass is R dependent and is $\mu(1+f) = \mu(R-2m)/R$; this implies that the point of closest approach is greater than the Schwarzschild radius, $2m$. Eqn.(19) may be solved using travelling molecular orbital methods ([5] and [11]). Setting in eqn.(19):

$$\Psi = \sum_{l=0}^\infty i^l(2l+1)\psi_l(R)P_l(\cos\Theta)\frac{S_l(R)}{R} \tag{20}$$

where the 'travelling' ($v = 0$) molecular orbitals ψ_l are given by

$$\left[-\frac{1}{2m_e}\nabla_\mathbf{r}^2 - \frac{Z_T e^2}{r_T} - \frac{Z_P e^2}{r_P} - \epsilon_l(R)\right]\psi(R) = 0. \tag{21}$$

We have ($m_e = 1$)

$$\left[\frac{d^2}{dR^2} + \frac{2\mu(R-2m)}{R}\{E - \epsilon_l(R) - \frac{m\mu c^2}{R} - \frac{(l+\frac{1}{2})^2}{R^2}\}\right]S_l(R) = 0. \tag{22}$$

In eqn.(20) Θ is the polar angle of R see eqn.(3). The region

$$R > 2m \tag{23}$$

corresponds to the exterior domain of Fiziev [12] and Matzner and Zamorano [13], the difference being that our gravitational force is linked to the heavy-particle electromagnetic forces which as stated above are partly acausal. Accordingly the potential in eqn.(22) corresponds to $s = 1$ and the Schwarzschild radius of $2m(\neq 1)$

in the potential $V_{S,l}(R)$ [12]. Coupled equations coupling the S_l are easily derived in the impact-parameter treatment [11] for calculational purposes.

The effective time of the eqn (1) is given to a first-order approximation by

$$\nu t = \pm\sqrt{R^2 - \rho^2} \tag{24}$$

given the equivalence of the wave and impact parameter treatments [14]. We have a consistent semiclassical treatment of quantum gravity, but bearing in mind that equations (9) and (19) contain acausality and noting that in the limit $M_P = 0 = M_T$, the normal time-independent Schrödinger equation obtains. We have considered a simple problem embracing spherical symmetry in the bound and collision aspects, given that these are massive particles with very slow relative motion. The general relativity of the black hole arises through our classical simulation. Our conjecture is that dark matter comprises ions, leptons and black holes and that dark energy comprises, in part, collisions between ions, leptons and black holes (cf [15]). In future papers our semiclassical quantum gravity treatment will be linked to the interior domain of [12] and possibly imaginary-frequency interior modes of black holes [13].

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Appendix A

In the normal wave treatment, the wave function is asymptotically given by:

$$\Psi \approx_{R\to\infty} e^{i\underline{k_0}\cdot\underline{R}} + \frac{e^{ik_0R}}{R}F(\hat{\underline{R}}) \tag{A1}$$

where F is the scattering amplitude. However in this paper

$$k_0 \to_{R\to\infty} k_0(1 - \frac{m}{R})$$

so that, given (see eg [5]: equation(4.377))

$$e^{i\hat{\underline{k}}\cdot\hat{\underline{R}}} \simeq_{R\to\infty} \frac{2\pi}{ikR}\left[e^{ikR}\delta(\hat{\underline{k}} - \hat{\underline{R}}) - e^{-ikR}\delta(\hat{\underline{k}} + \hat{\underline{R}})\right], \tag{A2}$$

we have:

$$\begin{aligned}\Psi e^{imk_0} \simeq \frac{2\pi}{ik_0R}\Big[&e^{ik_0R}\delta(\hat{\underline{k_0}} - \hat{\underline{R}}) \\ &- e^{-ik_0R}\delta(\hat{\underline{k_0}} + \hat{\underline{R}})e^{2mik_0}\Big] + \frac{e^{ik_0R}}{R}F(\hat{\underline{R}})\end{aligned} \tag{A3}$$

so that effectively the scattering amplitude is unchanged as is the impact parameter probability, the exception being perfect backward scattering.

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