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## Comment on “Mathematical and physical aspects of Kappa velocity distribution” [Phys. Plasmas 14, 110702 (2007)]

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A recent paper [L.-N. Hau and W.-Z. Fu, Phys. Plasmas 14, 110702 (2007)] deals with certain mathematical and physical properties of the kappa distribution. We comment on the authors’ use of a form of distribution function that is different from the “standard” form of the kappa distribution, and hence their results, *inter alia* for an expansion of the distribution function and for the associated number density in an electrostatic potential, do not fully reflect the dependence on  $\kappa$  that would be associated with the conventional kappa distribution. We note that their definition of the kappa distribution function is also different from a modified distribution based on the notion of nonextensive entropy. © 2009 American Institute of Physics. [doi:10.1063/1.3213388]

In a recent paper,<sup>1</sup> a number of mathematical aspects of the kappa velocity distribution<sup>2</sup> are presented. We are concerned that the results of Hau and Fu<sup>1</sup> do not correctly represent the full dependence on the parameter  $\kappa$  because the authors do not use the “conventional” kappa distribution<sup>2</sup> in their calculations.

The kappa distribution has been commonly used to fit particle data obtained in satellite-based experiments, for instance, as discussed by Refs. 2–8, as well as in some laboratory devices.<sup>9</sup> It describes a distribution function that has a Maxwellian-like core and a high-energy component of power-law form. The conventional isotropic, three-dimensional form of the kappa distribution function may be written concisely as

$$F_{\kappa}(v) = A_{\kappa} \left( 1 + \frac{v^2}{\kappa\theta^2} \right)^{-(\kappa+1)}. \quad (1)$$

This expression contains three parameters, viz.,  $A_{\kappa}$ ,  $\theta$ , and  $\kappa$ , the latter being a spectral index.<sup>2,10,11</sup> This is essentially the power index obeyed by the high-energy component of the equivalent energy distribution. Of these parameters, only  $\kappa$  is in a sense a free parameter, as the other two are constrained by the lowest (even) moments of the distribution function. Equating the lowest moment to the density  $N$  leads to

$$A_{\kappa} = \frac{N}{(\pi\kappa\theta^2)^{3/2}} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)}, \quad (2)$$

where  $\Gamma(x)$  is the usual gamma function. The parameter  $\theta$  is a characteristic speed, that has been termed an effective thermal speed by some authors. In fact, given the isotropic distribution function in the form of Eq. (1), it can easily be shown that  $\theta$  is the most probable particle speed. However,

we note that this is not the most probable speed that is found for a Maxwellian, i.e., what is usually termed the “thermal speed,”  $v_T = (2K_B T/m)^{1/2}$ , where  $T$  is the particle temperature,  $m$  the mass of the particles, and  $K_B$  is Boltzmann’s constant.

It is our contention that to properly relate  $\theta$  to the plasma temperature requires consideration of the second moment of the velocity distribution function and some care in its interpretation.

To illustrate the source of the divergent approaches to the use of the kappa distribution, we provide here a brief historical perspective.

The kappa distribution was first used as a fit to particle data found with the OGO-1 satellite by Vasyliunas,<sup>2</sup> following a suggestion by Olbert.<sup>2</sup> In that paper, the standard isotropic three-dimensional distribution was written as<sup>2</sup>

$$F_{\kappa}(v) = \frac{N}{w_0^3} \frac{\Gamma(\kappa+1)}{(\pi\kappa)^{3/2} \Gamma(\kappa-1/2)} \left( 1 + \frac{v^2}{\kappa w_0^2} \right)^{-(\kappa+1)}. \quad (3)$$

Here  $w_0$  was stated to be the most probable particle speed, in agreement with our statement regarding the equivalent  $\theta$  in Eq. (1).

Vasyliunas<sup>2</sup> also pointed out that it was often convenient to use the characteristic energy,  $E_0 = mw_0^2/2$ , corresponding to the characteristic speed  $w_0$ . This is the energy of the peak in the differential flux. By considering the second moment of the distribution function,

$$U = \int dv f(v) m v^2 / 2, \quad (4)$$

it was shown that the mean energy per particle is related to the most probable speed  $w_0$  through the associated energy  $E_0$  by

$$E_m \equiv U/N = \frac{3}{2} E_0 \left( \frac{\kappa}{\kappa-3/2} \right). \quad (5)$$

Finally, it was pointed out that in the special case  $\kappa \rightarrow \infty$ , the velocity distribution (3) reduces to the Maxwellian, for

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which equipartition of energy applies rigorously, and the expression (5) reduces to  $K_B T = E_0$ .

A few years later, Formisano *et al.*<sup>3</sup> used the kappa distribution (they called it a “ $K$  distribution”) to fit proton data measured in the magnetosheath. Substituting the above notation for theirs, we note that they introduced a “plasma temperature  $T$ ,” which they expressed as

$$K_B T = (1/2) m v_0^2 \left( \frac{\kappa}{\kappa - 3/2} \right). \quad (6)$$

Clearly, this is related to the mean energy per particle,  $E_m$ , of Vasyliunas, by  $E_m = (3/2) K_B T$ . Such a temperature definition, making use of equipartition of energy, although appropriate for the equilibrium distribution, the Maxwellian, is not strictly valid for a non-Maxwellian distribution. Nevertheless, there are practical advantages to using such an equivalent kinetic temperature, which can be a useful concept, and moreover, is accepted in practice for distributions that do diverge from the Maxwellian.<sup>12</sup>

In what may be the earliest theoretical paper based on the kappa distribution, Leubner<sup>13</sup> considered resonant wave particle interactions in a kappa plasma. Considering an anisotropic plasma ( $T_\perp \neq T_\parallel$ ), he wrote the kappa distribution function in a form equivalent to that of Vasyliunas, introducing  $\theta$  to represent the characteristic (most probable) speed. He also related the average energy to a temperature, hence obtaining

$$\theta^2 = \left( \frac{\kappa - 3/2}{\kappa} \right) \left( \frac{2K_B T}{m} \right). \quad (7)$$

Thus  $\theta$ , in Leubner’s early paper, is explicitly not the Maxwellian thermal speed  $v_T$ . Furthermore, it is clearly  $\kappa$ -dependent. This is true also of the equivalent characteristic speed  $w_0$  in Refs. 2 and 3.

Subsequently, a number of authors have studied various aspects and applications of the kappa distribution. Considering noise in a kappa plasma, Chateau and Meyer-Vernet<sup>14</sup> calculated “the equivalent temperature  $T$ ” from the mean particle energy, obtaining the expression

$$T = \frac{\langle m v^2 \rangle}{3K_B} = \frac{m v_0^2}{K_B} \frac{\kappa}{2\kappa - 3}, \quad (8)$$

where they used  $v_0$  in place of the notation  $w_0$  used by Vasyliunas,<sup>2</sup> and the distribution is normalized by setting  $\langle v^0 \rangle = 1$ .

It is important to note that the use of equipartition of energy implies that the kinetic temperature (defined through the second moment of the distribution function) is the same for all plasmas with the same mean kinetic energy per particle, independently of the exact form of the distribution function. In our case, it is thus independent of the value of  $\kappa$ , as one can see from Eq. (8). The second equality of Eq. (8) stresses the relationship between the temperature  $T$  and the most probable speed ( $v_0$ ) [i.e.,  $\theta$  in Eq. (7)] that is given explicitly in Eq. (7).

Independently, in a landmark set of papers, Summers, Thorne *et al.*<sup>10,11,15–19</sup> considered various waves in such a kappa plasma. They introduced a modified plasma dispersion

function  $Z_\kappa^*$ . This is the equivalent, for waves in a kappa-distributed plasma, of the usual  $Z$ -function<sup>20</sup> for a Maxwellian distribution. In particular, they used the notation of Leubner,<sup>13</sup> writing the isotropic, three-dimensional distribution function as<sup>10</sup>

$$F_\kappa(v) = \frac{N}{\pi^{3/2}} \frac{1}{\theta^3} \frac{\Gamma(\kappa + 1)}{\kappa^{3/2} \Gamma(\kappa - 1/2)} \left( 1 + \frac{v^2}{\kappa \theta^2} \right)^{-(\kappa+1)}, \quad (9)$$

where  $\theta = [(2\kappa - 3)/\kappa]^{1/2} (K_B T/m)^{1/2}$ , with  $T$  the “particle temperature.” The latter was defined in terms of the mean particle kinetic energy, as shown explicitly, for instance, for anisotropic plasmas.<sup>16,18</sup> We note that a well-defined value of  $\theta$  requires  $\kappa > 3/2$ .<sup>10,21</sup>

The formulation set out above has been followed by numerous papers on waves in kappa plasmas and related calculations, e.g., Refs. 7, 9, and 21–44 and many others.

From the above it also follows that one finds that the shielding distance in a  $\kappa$ -distributed plasma becomes  $\lambda_\kappa = \lambda_D [(\kappa - 3/2)/(\kappa - 1/2)]^{1/2}$ , where  $\lambda_D = (\epsilon_0 K_B T / n e^2)^{1/2}$  is the usual Debye length for a Maxwellian plasma having the same density and the same mean energy.<sup>6,14,25</sup>

As an aside, we note that integration over two velocity space coordinates leads to the one-dimensional kappa distribution function,<sup>10,27,38</sup> which can be written as

$$f_\kappa(v) = \frac{N_0}{(\pi \kappa \theta^2)^{1/2}} \frac{\Gamma(\kappa)}{\Gamma(\kappa - 1/2)} \left( 1 + \frac{v^2}{\kappa \theta^2} \right)^{-\kappa}, \quad (10)$$

where it is seen that the inverse power is  $\kappa$ , whereas for the three-dimensional distribution it is  $(\kappa + 1)$ .

Substituting in  $F_\kappa(v)$  for  $\theta$ , one may write the kappa distribution in an alternative form,<sup>32</sup> i.e.,

$$F_\kappa(v) = \frac{N_0}{(\pi \kappa \theta^2)^{3/2}} \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - 1/2)} \times \left( 1 + \frac{v^2}{[\kappa - 3/2][2K_B T/m]} \right)^{-(\kappa+1)}. \quad (11)$$

In some instances the “characteristic velocity” (the most probable speed) is loosely referred to as a “thermal velocity.” However, in all the above cases, one demands of the distribution function that the parameters  $A_\kappa$  and  $\theta$  should be obtained in a self-consistent manner from the lowest two even moments of the distribution, in which case it is clearly shown that, for a given value of mean particle energy, a measurable quantity,  $\theta$  must be a function of  $\kappa$ . As made explicit in Ref. 6, one is actually comparing kappa distributions having different values of  $\kappa$  with a Maxwellian counterpart having the same number density and energy density—what one may call “the equivalent Maxwellian.” Although one may have qualms about applying equipartition to a nonequilibrium distribution, the interpretation of  $T$  as being the temperature of the equivalent Maxwellian is valid and is useful, and it is accepted in practice. Sometimes this  $T$  is called the kinetic temperature.

We note that, as the factor  $(\kappa/[\kappa - 3/2]) > 1$  for finite  $\kappa$ , it follows from Eq. (8) that the “typical speed”  $v_0$  of the conventional kappa distribution is smaller than the Maxwellian thermal speed for the same value of temperature. From

that and from Eq. (9) it follows that the standard kappa distribution has a higher, narrower peak, and a broader base than the equivalent Maxwellian. This is intuitively reasonable—if the total energy content is the same, and the kappa distribution has more high-energy particles, then the velocity spread of the core, and hence the *perceived* thermal core temperature, of the kappa distribution must be lower than that of the equivalent Maxwellian. Clearly, however, from the definition of temperature, all distributions with the same mean energy per particle have the same temperature, as we have seen above.

In our discussion below we prefer not to discuss a “temperature” of the kappa distribution as such, but instead shall concentrate on distributions having the same average energy density (the second moment of the distribution), a measurable quantity, and write the temperature of the equivalent Maxwellian. We shall then explore the effects of different values of  $\kappa$  on various characteristics of the plasma.

It obviously makes practical and theoretical sense to compare the kappa distribution to a Maxwellian distribution that has the same average kinetic energy per particle. As we have seen, that leads to the result, Eq. (7), that the characteristic velocity (the most probable speed  $\theta$ ) is proportional to the thermal speed of the equivalent Maxwellian. The constant of proportionality between  $\theta$  and  $v_t$  is clearly dependent on  $\kappa$ . We reiterate this important point for future reference.

In contrast to the above formulation of the conventional kappa distribution, Leubner has, in a set of recent papers,<sup>45–47</sup> followed on the work of Silva *et al.*<sup>48</sup> to suggest a modified kappa distribution, which is based on the notion of nonextensive entropy. It assumes a simple transformation from a Tsallis distribution.<sup>49</sup> The latter is expressed in terms of a parameter  $q$ , and writing  $\kappa=1/(1-q)$ , Leubner<sup>45</sup> obtained a one-dimensional distribution [Eq. (6) of Ref. 45]. In particular, in it, the most probable particle speed  $\theta$  is replaced by the most probable speed of a Maxwellian, the thermal speed,  $v_t=(2K_B T/m)^{1/2}$ , where  $T$  is the temperature “of the species considered.” He then states that, by “analogy,” one finds a similar isotropic three-dimensional distribution [Eqs. (10) and (11) of Ref. 45],

$$F_L(\mathbf{v}) = \frac{N}{\pi^{3/2} v_t^3} \frac{1}{\kappa^{3/2}} \frac{\Gamma(\kappa)}{\Gamma(\kappa-3/2)} \left(1 + \frac{v^2}{\kappa v_t^2}\right)^{-\kappa}. \quad (12)$$

We note, importantly, that this three-dimensional function involves an inverse power  $\kappa$ , not  $(\kappa+1)$ . Later in the same paper, Leubner writes what he terms the “standard kappa distribution,” “written conventionally” in a form that is at first sight akin to that of Eq. (9), including an inverse power  $(\kappa+1)$ . However, unfortunately, he replaces  $\theta$  by the thermal speed  $v_t$ . He later states that an “effective thermal speed  $\theta=v_t[(\kappa-3/2)/\kappa]^{1/2}$  is commonly defined from the moments of the distribution function,” but does not use it. In a later paper,<sup>46</sup> similar results are put forward and lead to a distribution of the form of Eq. (12). In that paper, too, he writes of the “conventional” form of the kappa distribution function [alluding to that of Eq. (9)], but wrongly writes it in terms of the thermal speed  $v_t$  rather than the most probable speed  $\theta$ . In Ref. 47, a paper cited in the Response<sup>50</sup> to our Comment,

essentially the same approach is taken. That is, an equivalent of Eq. (12) appears, and the misleading representation (use of  $v_t$ ) of the conventional kappa distribution<sup>10</sup> persists.

Although what is in essence a “ $q$ -distribution,”<sup>49</sup> may possibly be regarded as a member of a broader kappa family of distributions, it seems incorrect to call it “the kappa distribution,” which is a well-established and much-used velocity/energy distribution function. In this context, we see that Bryant<sup>6</sup> showed a particularly good example of a data fit, for solar wind electrons, to a conventional kappa distribution. The distribution clearly has more particles in both the low-energy and high-energy regions than a Maxwellian does, with a dip in the vicinity of the effective thermal speed. This would not fit the modified distributions used by Leubner<sup>45–47</sup> and by Hau *et al.*,<sup>50</sup> which have fewer low-energy particles in a low- $\kappa$  distribution than is the case for the Maxwellian.

Turning now to the paper by Hau and Fu,<sup>1</sup> we note that the distribution function is defined as

$$f^\kappa(v) = \frac{N}{2\pi(\kappa v_\kappa^2)^{3/2}} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\frac{1}{2})\Gamma(\frac{3}{2})} \left(1 + \frac{v^2}{\kappa v_\kappa^2}\right)^{-(\kappa+1)}. \quad (13)$$

At first sight this is similar to our Eq. (9). However, although the characteristic speed  $v_\kappa$  appears from the notation (subscript  $\kappa$ ) to have kappa dependence as  $\theta$  does, it is, in fact, defined as  $v_\kappa=[(2\kappa-3)/\kappa \times K_B T_\kappa/m]^{1/2}$ , where “the characteristic temperature  $T_\kappa$  is  $T_\kappa=\kappa T/(\kappa-3/2)$  with  $T$  being the temperature in the Maxwellian distribution,  $f=N(m/2\pi K_B T)^{3/2} \exp(-mv^2/2K_B T)$ .” By substituting for  $T_\kappa$  and hence for  $T$  in  $v_\kappa$ , one sees that this definition of the characteristic speed is circular, and that the denominator in Eq. (13) then simply reduces to  $2\kappa K_B T/m = \kappa v_t^2$ , i.e.,  $v_\kappa \equiv v_t$ , the Maxwellian thermal speed. Thus, in this formulation, there is no need to introduce an apparently kappa-dependent characteristic speed  $v_\kappa$ .

The introduction of the notation  $T_\kappa$  in Ref. 1 appears to follow the notation of Maksimovic *et al.*,<sup>51</sup> a paper cited in the Response to our Comment. However, the latter paper does not define  $T_\kappa$  in the form given in Ref. 1. Instead, it is given as  $T_\kappa=m\langle v^2 \rangle/3K_B$ , as in Eq. (8), i.e., they<sup>51</sup> use the same form for the temperature as we have discussed above.<sup>3,14</sup> As we have seen, that agrees with the form used in Ref. 10 and the many papers that follow it.

In the Response<sup>50</sup> to our Comment, the authors to some extent clarify their intention regarding  $v_\kappa$ , by explicitly writing the distribution in terms of the thermal velocity of a Maxwellian, rather than in terms of  $v_\kappa$ ,

$$f^\kappa(v) = \frac{N}{2\pi(\kappa v_{\text{th}}^2)^{3/2}} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)\Gamma(\frac{3}{2})} \left(1 + \frac{v^2}{\kappa v_{\text{th}}^2}\right)^{-(\kappa+1)}, \quad (14)$$

with  $v_{\text{th}}$  defined through  $v_{\text{th}}^2=2K_B T/m$ .

In principle, it does seem to be surprising, and indeed illogical, that a non-Maxwellian distribution should be defined *a priori* in terms of an unknown Maxwellian thermal speed. The result is that, in considering different values of  $\kappa$ , the authors are comparing different “equivalent temperatures” associated with different values of kappa with the temperature of a specific preordained Maxwellian. It is not clear



how that temperature is defined and to be measured. Bearing in mind the uncertainty of the meaning of a temperature for a non-Maxwellian (“nonequilibrium”) distribution, that would seem to be an exercise fraught with difficulties in interpretation. We believe that the standard approach, as we have outlined above, is both more satisfying theoretically and relates to the observable quantity, the mean kinetic energy per particle.

Furthermore, in using the thermal speed rather than the most probable speed  $\theta$ , the authors<sup>1</sup> may appear to be following the model of Leubner.<sup>45–47</sup> However, as we have spelt out above, the latter papers use a distribution in which the inverse power is  $\kappa$ , while in the authors’ representation<sup>1</sup> it is  $(\kappa+1)$ . Thus, the form of distribution function used by Hau and Fu is different from both the conventional form [Eq. (9)] and the Tsallis-like form [Eq. (12)].

Following on from their distribution [Eq. (13)], the authors obtain an expansion<sup>1</sup> of a normalized form of  $f^\kappa(v)$  in the form of a power series in  $(x/\kappa)$ , where  $x=(v/v_\kappa)^2$ , as given in their Eqs. (3) and (4). However, as we have seen,  $x$  itself includes  $\kappa$ -dependence if  $v_\kappa$  is correctly defined, as given for  $\theta$  above. Thus the expansion given by Hau and Fu<sup>1</sup> neglects dependence on  $\kappa$  that a proper definition of  $x$  would provide. One can see immediately from the formulation given in our Eq. (10) above that to rectify this it is important to write the expansion not in  $(x/\kappa)$ , but in  $(X/[\kappa-3/2])$ , where  $X=v^2/(2K_B T/m)=(v/v_\kappa)^2$  is a variable that is independent of  $\kappa$ . We note that  $X=x(\kappa-3/2)/\kappa$ . This obviously has mathematical (and hence physical) implications down the line, such as the range of  $v$  over which the expansion is valid, for distributions with a very high superthermal content, i.e., for, say,  $5/2 > \kappa > 3/2$ .

Later in the paper,<sup>1</sup> the authors calculate moments of their distribution function to find expressions for the density and hence pressure of a species designated as  $\alpha$ , in an electrostatic potential  $\varphi$ , to obtain

$$n_\alpha = N \left( 1 + \frac{q_\alpha \varphi}{\kappa K_B T_\alpha} \right)^{-\kappa+1/2}, \quad (15)$$

in their notation. We stress that, as the denominator does not represent the full dependence of  $\theta$  on  $\kappa$ , Eq. (15) suppresses some of the  $\kappa$ -dependence of the density in a potential  $\varphi$ . This is particularly important for low values of  $\kappa$ , approaching  $(3/2)$ . This construction is carried through from Eqs. (12) to (17) of their paper.<sup>1</sup>

In fact, the full dependence of the density function on  $\kappa$  is reflected in our recent papers,<sup>32,33</sup> where we have written (in equivalent notation) the density for electrons and ions in an electrostatic potential  $\varphi$  explicitly as

$$n_\alpha(\varphi) = N_\alpha \left( 1 \pm \frac{2e\varphi}{m_\alpha \kappa_\alpha \theta_\alpha^2} \right)^{-(\kappa_\alpha-1/2)}, \quad (16)$$

with  $\theta_\alpha^2 = [(\kappa_\alpha-3/2)/\kappa_\alpha][2K_B T_\alpha/m_\alpha]$  and  $\alpha=e, i$  for the respective species. In this expression the upper (lower) sign gives the expression for the ions (electrons) and  $T_\alpha$  is the temperature of the equivalent Maxwellian, as discussed above. Substitution for  $\theta_\alpha$  then reveals the full dependence of the density on  $\kappa_\alpha$ , viz.,

$$n_\alpha(\varphi) = N_\alpha \left( 1 \pm \frac{e\varphi}{[\kappa_\alpha-3/2]K_B T_\alpha} \right)^{-(\kappa_\alpha-1/2)}. \quad (17)$$

Naturally, this expression is obtained from either the three-dimensional or the one-dimensional form of the kappa distribution function.<sup>32</sup> Appropriate normalized forms may then be deduced for each species,<sup>32</sup> characterized by the factor  $(\kappa_\alpha-3/2)$  in the denominator.

Comparing Eq. (17) with that obtained by Hau and Fu<sup>1,50</sup> [Eq. (15)], one sees that the latter does not reflect the rapid changes that occur for  $\kappa$  values approaching  $(3/2)$ .

Finally, we note that in their Response<sup>50</sup> the authors cite a paper by Collier.<sup>52</sup> However, a careful reading of that work shows that Collier used the form of kappa distribution given in Ref. 10, i.e., our Eq. (9), and then indicated that one normally defines a temperature for this non-Maxwellian distribution in the form that we have given above [Eq. (8)]. Collier then argued that in evaluating such a temperature from data, there are at times difficulties in fitting the low-flux high-energy part of the spectrum, resulting in the fit often being dominated by the core Maxwellian part. On that basis he introduced a second temperature  $T_{\text{core}}$ , associated with that Maxwellian core. Interestingly, he found that for ion distributions  $T_{\text{core}}$  increases with increasing  $\kappa$ , a result which is in line with our earlier discussion concerning the narrowness of the “core” distribution of a kappa distribution.

In related papers,<sup>53,54</sup> the kappa distribution is again presented in the conventional form, Eq. (9), although the core temperature is also introduced in the discussion. We also note that a close examination of Fig. 2 of Ref. 55 indicates that the calculated  $\kappa$  distribution found by Collier in that case appears to have a central peak that is higher than that of the Maxwellian, i.e., it agrees better with the shape obtained from Eq. (9) than with that from the alternative distribution, as given in Fig. 1 of the Response.<sup>50</sup>

In summary, we agree that Hau and Fu<sup>1</sup> have found a number of useful mathematical expressions for a non-Maxwellian distribution that, like a kappa distribution, has a high-energy tail. However, we have shown that in their calculations they have used neither the standard, conventional kappa distribution [Eq. (9)] (Refs. 2, 10, and 13) nor the modified ( $q$ -like) distribution [Eq. (12)] recently suggested,<sup>45</sup> that is based on nonextensive entropy. Further we have shown that as a result of an unusual definition of  $T_\kappa$ , aspects of their original presentation<sup>1</sup> involve a circular argument, as a result of which their introduction of  $v_\kappa$  is superfluous. We have drawn attention to the misleading nature of the expansion that they obtain for low velocities, in that it suppresses an important part of the dependence of the kappa distribution function on  $\kappa$ , and we have given the density of a  $\kappa$ -distributed component in an electrostatic potential, in a form in which its full dependence on  $\kappa$  is reflected.

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